## THE FOURTEENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

MANY members of the Society recall with pleasure the summer meeting and colloquium held at Ithaca in 1901. After an interval of six years the summer meeting, the fourteenth of the series, was again convened at Cornell University on Thursday and Friday, September 5–6, 1907. By close economy of time the scientific proceedings were condensed into two sessions on Thursday and a morning session on Friday. Friday afternoon was devoted to an excursion on Lake Cayuga, Mr. H. H. Westinghouse, of the University, having kindly placed his steam yacht at the disposal of the members. The evening gatherings at the Town and Gown Club also furnished pleasant opportunities for social intercourse.

The first session opened with an address of welcome by Professor Wait. At the close of the meeting resolutions were adopted expressing the Society's appreciation of the generous hospitality of the University and its officers.

The attendance included the following forty-seven members of the Society:

Professor O. P. Akers, Professor G. A. Bliss, Mr. W. C. Brenke, Professor W. G. Bullard, Dr. W. B. Carver, Professor F. N. Cole, Professor L. L. Conant, Professor E. W. Davis, Professor S. C. Davisson, Dr. E. L. Dodd, Dr. F. J. Dohmen, Professor W. P. Durfee, Professor L. P. Eisenhart, Professor H. B. Fine, Professor W. B. Fite, Professor W. B. Ford, Professor A. S. Gale, Professor C. N. Haskins, Professor E. R. Hedrick, Professor J. I. Hutchinson, Dr. L. C. Karpinski, Professor O. D. Kellogg, Professor T. E. McKinney, Professor James McMahon, Professor W. H. Metzler, Mr. E. A. Miller, Professor G. A. Miller, Dr. R. L. Moore, Mr. F. W. Owens, Dr. Arthur Ranum, Professor E. D. Roe, Professor D. A. Rothrock, Dr. F. R. Sharpe, Miss M. E. Sinclair, Dr. C. H. Sisam, Professor P. F. Smith, Professor Virgil Snyder, Professor H. F. Stecker, Professor J. H. Tanner, Professor Anna L. Van Benschoten, Professor E. B. Van Vleck, Professor Oswald Veblen, Professor L. A. Wait, Dr. W. D. A. Westfall, Professor H. S. White, Professor J. M. Willard, Professor Alexander Ziwet.

The President of the Society, Professor H. S. White, occupied the chair, being relieved by Professors Fine and E. B. Van Vleck. The Council announced the election of the following persons to membership in the Society: Mr. Thomas Buck, University of Chicago; Mr. Arnold Dresden, University of Chicago; Mr. T. H. Hildebrandt, University of Chicago; Mr. W. J. King, Harvard University; Mr. J. O. Mahoney, High School, Dallas, Texas; Dr. J. F. Messick, Randolph-Macon College; Mr. H. W. Powell, College of the City of New York. Six applications for membership in the Society were received. The total membership is now 569.

The following papers were read at the summer meeting:

(1) Professor L. E. DICKSON: "Modular theory of group matrices."

(2) Professor W. B. FORD: "Sur les équations linéaires aux différences finies."

(3) Professor R. D. CARMICHAEL: "On the classification of plane algebraic curves possessing fourfold symmetry with respect to a point."

(4) Professor R. D. CARMICHAEL: "Note on certain inverse problems in the simplex theory of numbers."

(5) Dr. W. B. CARVER: "The ten special $\Gamma^4_{6,2}$ configurations in the Pascal hexagram."

(6) Professor E. O. LOVETT: "Generalization of a problem of Bertrand in mechanics."

(7) Professor E. O. LOVETT: "The invariants of a group which occurs in the problem of $n$ bodies."

(8) Professor E. R. HEDRICK: "A peculiar example in the theory of surfaces."

(9) Professor E. R. HEDRICK: "A smooth closed curve composed of rectilinear segments."

(10) Professor R. D. CARMICHAEL: "On certain transcendental functions defined by a symbolic equation."

(11) Dr. D. C. GILLESPIE: "On the canonical substitution in the Hamilton-Jacobi canonical system of differential equations."

(12) Professor G. A. MILLER: "The invariant substitutions under a substitution group."

(13) Professor G. A. MILLER: Methods of determining the primitive roots of a number."

(14) Professor VIRGIL SNYDER: "On a special algebraic curve having a net of minimum adjoint curves."

(15) Professor JAMES MCMAHON: "The differential geometry of the vector field. Second paper: lamellar field."

(16) Professor L. E. DICKSON: "Commutative linear groups."

(17) Professor L. E. DICKSON: "A simple derivation of the canonical forms of linear transformations."

(18) Professor EDWARD KASNER: "Geometric interpretation of integrating factors."

(19) Professor EDWARD KASNER: "The conformal representation of geodesic circles."

(20) Mr. A. R. SCHWEITZER: "On the relation of right-handedness in geometry."

(21) Dr. F. L. GRIFFIN: "On the law of gravitation in the binary systems, II."

(22) Dr. F. L. GRIFFIN: "Certain trajectories common to different laws of central force."

(23) Professor E. W. DAVIS: "Colored imaginaries. I, Imaginaries in the plane."

(24) Professor E. W. DAVIS: "Colored imaginaries. II, Imaginaries in space."

(25) Dr. C. H. SISAM: "On the equations of quartic surfaces in terms of quadratic forms."

(26) Professor VIRGIL SNYDER: "On the range of birational transformation of curves having genus greater than that of the canonical form."

(27) Professor G. A. MILLER: "Third report on recent progress in the theory of groups of finite order."

(28) Professor OSWALD VEBLEN: "Continuous increasing functions of ordinal numbers."

(29) Professor H. S. WHITE and Miss K. G. MILLER: "Note on Lüroth's type of plane quartic curve."

(30) Professor W. B. FITE: "Concerning the degree of an irreducible linear homogeneous group."

(31) Dr. ARTHUR RANUM: "Concerning linear substitutions of finite period with rational coefficients."

(32) Professor R. P. STEPHENS: "Certain curves of class $n$ having $n-2$ tangents in any given direction."

(33) Professor ANNA L. VAN BENSCHOTEN: "Curves of genus 4 which remain invariant under birational transformation."

(34) Miss M. E. SINCLAIR: "On a discontinuous solution in the problem of the surface of revolution of minimum area."

(35) Dr. MAURICE FRÉCHET: "Sur les opérations linéaires (troisième note)."

(36) Professor A. G. GREENHILL: "The elliptic integral in electromagnetic theory."

Dr. Fréchet's paper was communicated to the Society through Professor E. B. Van Vleck. In the absence of the authors, Dr. Gillespie's paper was presented by Professor Snyder, Dr. Griffin's papers by Professor Veblen, Professor Greenhill's paper by Professor McMahon, and the papers of Professor Dickson, Professor Carmichael, Professor Lovett, Professor Kasner, Mr. Schweitzer, Professor Stephens and Dr. Fréchet were read by title.

Professor Dickson's first paper was published in the July number of the *Transactions*. Professor Miller's first paper appeared in the October BULLETIN. Professor Miller's report, Professor Snyder's first paper, and Professor Carmichael's second paper are published in the present number of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

2. Professor Ford's paper considers the linear difference equation

$$a_0(x)\Delta^n y + a_1(x)\Delta^{n-1} y + \cdots + a_n(x)y = 0,$$

and obtains certain results analogous to those established during recent years by Dini for the linear differential equation

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n(x)y = 0.$$

Four general theorems are established concerning the existence and character of the solutions of the above difference equation, the last three of which are especially adapted to the study of the solutions $y(x)$ in the neighborhood of the point $x = \infty$. The results obtained find noteworthy application in the study of the convergence of algebraic continued fractions, as will appear more at length subsequently. The paper appeared in the last number (volume 13, number 4) of the *Annali di Matematica*.

3. Professor Carmichael considers the classification of plane algebraic curves possessing fourfold symmetry with respect to a point. The discussion is a generalization of his previous paper on quartic curves possessing the same property. It results in the following theorem:

Excluding curves consisting of isolated points only, the only loci of the $n$th order possessing fourfold symmetry with respect to a point taken as the origin are those whose rectangular cartesian equations are of one of the forms

$$\Sigma[a_u(x^t y^s + (-1)^t x^s y^t)] = 0, \quad \Sigma[a_u(x^t y^s - (-1)^t x^s y^t)] = 0,$$

where $n$ must be even and $s = 0, 1, 2, \cdots, n$; $t = s, s+1, \cdots, n$; subject to the condition that $t + s = 0, 2, 4, \cdots, n$.

The paper will be published in the *Annals of Mathematics*.

5. In his papers, "Sur quelques théorèmes, etc.," *Crelle's Journal*, volume 31 (1846), and "On Pascal's theorem," *Quarterly Journal*, volume 9 (1868), Cayley called attention to a peculiar special case of the configuration $\Gamma^4_{6,2}$ which occurs ten times in the Pascal hexagram. In a paper "On the Cayley-Veronese configurations," *Transactions*, volume 6 (1905), Dr. Carver showed that there is a pencil of six conics connected with any $\Gamma^4_{6,2}$. The pencils connected thus with the special $\Gamma^4_{6,2}$'s in the Pascal hexagram are studied in the present paper, and their relation to the fundamental conic of the hexagon is shown. The notation used is defined in the author's paper cited above.

6. In the plane $xoy$ let a point $(x, y)$ be acted upon by an acceleration force whose components are certain functions of the coordinates of the point. Professor Lovett proposes to examine in what cases all the trajectories, whatever be the initial conditions, are curves of the family defined by the differential equation

$$C(\alpha, \beta, \gamma, \delta, \epsilon) \equiv \alpha(12)^2(15) + \beta(12)^2(24) + \gamma(12)(13)(14) + \delta(13)^3 + \epsilon(12)(13)(23) = 0,$$

where $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$ are constants, and $(i, j)$ is written for $x^{(i)}y^{(j)} - x^{(j)}y^{(i)}$, the indices indicating derivations with regard to any independent variable. If the left member is $C(9, 45, -45, 40, -90)$, we have the form in which Stéphanos has written the differential equation of the conics first given by Monge.

This problem was studied by Bertrand * in the case where

* Bertrand: "Note sur un problème de la mécanique," *Comptes rendus*, vol. 84, p. 731. Bertrand had formulated the problem in a previous note "Sur la possibilité de deduire d'une seule les lois de Kepler et le principe de l'attraction," ibid., p. 673.

the force has in general a unique direction at every point and the trajectories are conic sections. Bertrand showed that in this case the force ought either to pass through a fixed point or be parallel to a fixed direction. Stéphanos * has recently examined the general case of conic trajectories when the force has not necessarily a unique direction; he finds that the problem admits of no other solution than those included in the case considered by Bertrand, and shows that every force under which a point describes a conic section, whatever be the initial conditions, always passes through a fixed point or remains parallel to a fixed direction. Finally by applying his method to the determination of the explicit forms of the forces in the case of conics Stephanos rediscovers the known theorems of Darboux and Halphen.†

In the present note free use is made of the method and formulas developed by Stéphanos in his memoir to which reference has been made. The analytic conditions determining the forces are set up and transformed with a view to make appear the condition for central forces or forces parallel to a fixed direction; it results that this condition may be satisfied and one of the parameters in the above differential equation remain absolutely arbitrary, that is, if $\alpha$, $\gamma$, $\delta$ are in the ratios $9 : -45 : 40$, and $\epsilon$ equals $-2\beta$, the forces under which a point describes the corresponding curve, whatever be the initial conditions, either pass through a fixed point or remain parallel to a fixed direction. Furthermore in this case, which obviously includes the conic sections, the differential equations to be integrated for the forces are identical in form to those encountered by Stéphanos; his results and those of Darboux and Halphen thus become applicable to this more general problem.

7. If in the problem of the motion of $n$ bodies about a fixed center under forces varying as an arbitrary function of the distance we seek those integrals which do not involve the law of force,‡ we come upon a system of linear partial differential equations formed by equating to zero $n(2n+1)$ infinitesimal

* Cyparissos Stéphanos: "Sur les forces donnant lieu à des trajectories coniques," *Crelle's Journal*, vol. 131 (1906), pp. 136–151.

† Darboux, Halphen, *Comptes Rendus*, vol. 84, pp. 760, 939.

‡ Gravé seems to have been the first to seek integrals of this kind in the problem of three bodies, and he found that the differential equations in the Bour-Bertrand form admit of no integrals independent of the law of force other than those already known. See his paper: "Sur le problème des trois corps," *Nouv. Ann. de Math.*, series 3, vol. 15 (1896), pp. 537–547.

transformations. That these infinitesimal transformations generate a continuous group was pointed out by Professor Lovett in an unpublished paper previously presented to the Society (December 29, 1902); in the present paper the methods of Sophus Lie are employed to construct the invariants of this group.

8. In Professor Hedrick's first paper an example of a surface is given in which the distance from a point on the normal to each normal section is a minimum, while the distance from the same point to the surface itself is not a minimum. The example shows the fallacy of a geometric conclusion sometimes assumed as obvious, and reference is made to a recent use of the fallacious theorem. (BULLETIN, volume 13, No. 9, page 447.)

9. In this note, which will be offered for publication to the *Annals of Mathematics*, Professor Hedrick gives an example of a closed curve formed by the junction of (an infinite number of) rectilinear segments, which is nevertheless *smooth*, in the sense defined by Professor Osgood (Funktionentheorie). The statement is to be understood in the sense that between any two points on the curve there is an arc which is a segment of a straight line. The example is obtained by simple integration from a well-known case; it seems worth while in illustrating the non-intuitive character of the ideas involved.

10. In Professor Carmichael's third paper a large class of transcendental functions is defined in the following manner: Let $u = f(x)$ and $v = f(y)$, and put $u^v = v^u$. This last equation has a solution $u = f(x) = \mu^{1/(\mu-1)}$, $v = f(y) = \mu^{\mu/(\mu-1)}$, where $\mu$ is a variable. Solving these functional equations, we have $x = g(\mu^{1/(\mu-1)})$ and $y = g(\mu^{\mu/(\mu-1)})$, where $g$ is the function inverse to $f$. These equations define a function $y$ of $x$. It is this function which is studied in the paper. Attention is confined chiefly to questions of continuity and development in series.

11. Lie established a one to one correspondence between the integrals of the canonical system of differential equations and the one-parameter continuous groups of contact transformations of which the system admits: *i. e.*, making use of an integral,

one can construct the infinitesimal transformation belonging to a group of which the system admits, and conversely by making use of the infinitesimal transformation one can construct an integral of the system.* This theorem is the foundation of the modern transformation theorems of dynamics.†

The single canonical substitution on the other hand has served chiefly in the simplification of the equations.‡ The paper of Dr. Gillespie defines a single canonical substitution, the knowledge of which leads immediately to the construction of an integral of the system.§

13. Professor Miller's note aims to exhibit some elementary relations between well known methods of finding the primitive roots of a number and the properties of a cyclic group. It also contains a very elementary proof of the following theorem, which is believed to be new: When the exponent to which the primitive roots of a given odd number $g$ belong is of the form $2q$, $q$ being any odd prime, each of the primitive roots of $g$ is given once and only once by $-\alpha^2$, $1 < \alpha < g/2$ and $\alpha$ being prime to $g$. When $g$ is even, the primitive roots are obtained by replacing the even values of $\alpha^2$ by $g/2 - \alpha^2$, while the odd values remain unchanged. In the former case $\alpha^2$ belongs to exponent $q$, while $\alpha^2$ or $g/2 + \alpha^2$ belong to this exponent for odd or even values of $\alpha$, respectively, when $g$ is even and has primitive roots. From this we derive the following useful corollary: Every prime of the form $2q + 1$, $q$ being any odd prime, has for its primitive roots $-\alpha^2$, $1 < \alpha < q + 1$. It is also pointed out that the theorem which affirms that the product of all the primitive roots of any number which has more than one such root is congruent to unity is a special case of the theorem that the product of all the operators of the same order $> 2$ in any abelian group is identity.

15. Professor McMahon's paper considers the field of a vector whose three components at any point are the partial deriva-

* Whittaker, Analytical dynamics, p. 308.

† l. c., p. 292.

‡ Jacobi, Vorlesungen über Dynamik. Poincaré, Mécanique céleste, vol. 1. For condition that a substitution be canonical, see Lie, *Norw. Ark. für Math.*, etc., 1877. Also Jacobi, Vorlesungen über Dynamik.

§ For the two theorems upon which the argument depends, see the author's thesis, "Anwendungen des Unabhängigkeitssatzes auf die Lösung der Differentialgleichungen der Variationsrechnung," p. 30; Göttingen, 1906. Also, "On the construction of an integral of Lagrange's equations in the calculus of variations," BULLETIN, vol. 13 (1907), pp. 345–348.

tives of a given function of the coordinates of the point. Among the topics treated are the following: Derivative of the vector for any direction in its field; variation of derivative; variation of tensor; rotation of the field vector for displacement in any direction; variation of rotation; lines of no rotation; family of surfaces orthogonal to the vector lines; relation of the indicatrix conic at any point to the vector field around the point; invariants of the field.

16. In the second paper by Professor Dickson it is shown at the outset that every maximum (*i. e.*, not contained in a larger) commutative group $M$ of $m$-ary linear homogeneous transformations $(a_{ij})$ in a field $F$ may be completely defined by linear homogeneous relations between the $a_{ij}$ with constant coefficients belonging to $F$. In other words, the independent parameters of $M$ enter linearly and homogeneously. Continuous commutative linear groups with the last property have been considered by Professor W. Burnside, *Proceedings of the London Mathematical Society*, volume 29 (1898), pages 325-352, chiefly from the standpoint of the invariant factors of the characteristic determinant of the general transformation of the group. He treats in detail the case of two invariant factors distinct from unity; but (page 341) draws an erroneous conclusion (that $b_1 = 0$, whereas $a_1 - e_1$ is zero in view of the initial hypothesis), which renders invalid the first step, as well as all the successive steps, of his reduction process. Moreover, of the groups thus omitted, there occur examples with $b_1 \neq 0$ which are not reducible to groups with $b_1 = 0$ and of the type considered. Burnside starts with a particular transformation $S$ (in canonical form) with the desired invariant factors, then exhibits the most general linear transformation $T$ commutative with $S$, and finally attempts to select and normalize all the transitive commutative groups composed exclusively of transformations $T$. He expected to prove that all such groups have the same invariant factors as the individual transformation $S$. It is clear *à priori* that this will be true only in trivial cases. A correct version of Burnside's methods of constructing commutative linear groups would therefore still have the objection of being highly redundant. To establish the non-equivalence of two groups, Burnside makes use (§11) of another classification (§9), based upon a highly desirable normal form of the commutative linear group. In this normal form, at least one variable $x_i$ is merely

multiplied by a constant, at least one $y_j$ is replaced by $by_j + \Sigma a_i x_v$ etc. As Burnside omits references, it should be said that this normal form was given implicitly by Lie in 1878 (cf. Lie-Engel, Transformationsgruppen, I, page 589), and explicitly for commutative linear groups of order a power of $p$ in the $GF[p^r]$ by Jordan in 1870 (Traité des substitutions, page 147). However, the latter author makes errors of various kinds (on page 148) in determining the commutative groups of the simplest types, as is pointed out in detail in the present paper. By means of the Jordan normal form (established for a general field $F$) and the theorem stated at the outset, it is a comparatively simple algebraic problem to determine explicitly all the maximum commutative linear groups. The more difficult problem of their reduction to non-equivalent types is found to depend upon the normalization of families of quadratic forms $\Sigma c_i q_i$ by linear transformation both of the parameters $c_i$ and the variables in $q_i$. The paper will be offered for publication in the *Annals of Mathematics*.

17. The third paper by Professor Dickson gives a direct elementary proof of Jordan's canonical form of any linear homogeneous transformation (Traité des substitutions, pages 114–126, for the case of integral coefficients reduced modulo $p$, a prime), as extended to the general case of coefficients in an arbitrary field $F$ (*American Journal*, 1902, pages 101–108). Let the characteristic equation have the roots $\alpha, \beta, \cdots$, with the multiplicities $a, b, \cdots$. The plan of the proof is to introduce $a$ variables depending on $\alpha$, $b$ variables depending on $\beta, \cdots$, and to show that the new variables are independent. As the various sets are here introduced independently and not in succession, one avoids the necessity of the numerous normalizations made by Jordan (pages 120–123) and the intricate proof (page 121) that each set involves a single irrationality. The essential value of the canonical form lies less in the form of the explicit formulas than in the exact relation of the new variables to the irrationalities introduced. The paper will be offered for publication in the *Annals of Mathematics*.

18. In this note Professor Kasner separates the geometric from the analytic elements involved in Lie's well-known interpretation, and obtains a concrete representation by introducing a surface whose contour lines are the curves defined by the

differential equation. The slope of this surface visualizes the integrating factor.

19. In a former paper Professor Kasner showed that when a surface is represented conformally on a plane its geodesics are pictured by a family of curves whose central loci are straight lines. In the case of curves of given geodesic curvature it is shown that the corresponding loci are conics. When the curvature is varied the conics corresponding to a given point have a common focus and a common directrix.

20. With reference to a remark made by Professor Study in the April, 1907, number of the *American Journal of Mathematics* (page 101, note) Mr. Schweitzer observed that he has already shown that right-handedness is a descriptive relation which, with the aid of the indefinable point and ordered dyad, generates a descriptive geometry extensible, on the basis of the axioms, to the usual spatial projective geometry by well known methods.*

Mr. Schweitzer then briefly indicated some of the more important results of his researches in descriptive, projective and metrical (euclidean) geometry, which will be offered to the *American Journal of Mathematics* for publication in detail:

(1) There exists not only a fundamental linear relation (viz., the relation of Vailati) but a fundamental $n$-dimensional descriptive relation $R_n$ $(n = 1, 2, 3, \cdots)$. For $n = 3$ the relation is that of absolute right-handedness. Further, the relation $R_n$ is $n$-dimensionally transitive, (*i. e.*, $xR_n a_2 \cdots a_{n+1}$, $a_1 R_n x a_3 \cdots a_{n+1}$, $\cdots a_1 R_n a_2 a_3 \cdots x$, imply $a_1 R_n a_2 \cdots a_{n+1}$) and alternating.

(2) Descriptive or projective order in $n$-dimensions $(n = 1, 2, 3, \cdots)$ may be generated $p$-dimensionally $(p = 1, 2, 3, \cdots n)$, so that the geometric spaces may be said to have the double index $(p, n)$, $1 \leqq p \leqq n$. The preceding results have an important critical bearing on the discussion of geometric order given by Russel in his Principles of Mathematics.

(3) The descriptive $n$-dimensional systems generated by the relation $R_n$ may be transformed into equivalent systems generated by the relation $K_n$,† the latter for $n = 3$ express-

* The relation of right-handedness referred to is the relation $R_3$ of the author, cf. BULLETIN, Nov., 1906, p. 79. The results indicated above were also demonstrated by the author in a paper read at the University of Chicago, Feb., 1905; at that time an undefined *class*, corresponding to the relation $R_3$, was used.

† Cf. BULLETIN, June and Nov., 1906.

ing concretely sameness of sense, or relative right-handedness. For $n \geqq 3$ Mr. Schweitzer showed that the corresponding $n$-dimensional $K_n$ systems may be made "complete" for euclidean geometry by adding a certain number of axioms, the system of real numbers being introduced in a very simple manner. These "complete" systems constitute an elegant basis for Grassmann's Ausdehnungslehre and the exposition of the latter by Peano. In this connection it is of interest to consider a $K_0$ system of dimensionality zero, in addition to the $K_n$ systems mentioned above.

21. In an earlier paper (read December 28, 1906), Dr. Griffin showed that Newton's law of attraction is the only one which permits the ellipse as a central orbit, and which satisfies the conditions: (1) The force is a single-valued function of the distance; (2) real motion is possible in all parts of the plane; and (3) there is no circle about the center of force, throughout which the force continually decreases toward the center. It has since been found that condition (1) is superfluous, as any law not satisfying (1) fails also to satisfy (2). In the present paper proof of this redundancy is given; and as a corollary thereto is obtained the theorem of Darboux and Halphen. *

22. It has been generally overlooked that the law of force, under which a given curve is described as a central orbit with a given constant of areas, cannot be uniquely determined from a knowledge of the position of the center of force. Dr. Griffin, treating this question in his second paper, considers especially a few of the infinitude of laws, other than that of Newton, for which a possible orbit is the ellipse with the center of force at a focus. Various properties of other trajectories than the original ellipse are pointed out.

23. This paper is the development of a communication made by Professor Davis to the Chicago section at its March meeting (see BULLETIN, June, 1907, page 436). In it are presented in order the notation with its adaptation to homogeneous coordinates and the effect of real and imaginary shifts of the axes; scalar and vector operations; the linear relation — central, non-central, and real; the quadratic relation as exhibited by the real and the imaginary circle and by other conics; the real cubic and its nine inflexions.

* *Comptes rendus*, vol. 84, pp. 760–762, and 936–941. Cf. also Tisserand, Mécanique céleste, vol. 1, pp. 36–42.

24. In this second paper, Professor Davis applies his method to three-dimensional space. He considers the planar relation, axial, non-axial but spatial, non-spatial; the linear relation, spatial and axial, spatial but not non-axial; combination of linear and planar relations; the cylinder; the sphere; the connection with Von Staudt's representation of imaginaries by involutions.

25. Dr. Sisam showed that the quartic surface $\phi(A, B, C, D) = 0$, $\phi$, $A$, $B$, $C$ and $D$ being quaternary quadratic forms, which was considered by Durrande * in the *Nouvelles Annales* as the most general quartic surface, is not the most general surface of fourth order, but that a single condition must be satisfied by the coefficients. He also showed that the equation of the most general quartic surface can be written in the form $\theta(A, B, C, D, E) = 0$.

26. In Professor Snyder's second paper it is shown that if a plane curve of order $n$ has not more than $2(n-4)$ double points, the simplest curve to which it can be birationally reduced can be obtained by a Cremona transformation. The maximum number of basis points of a net of adjoint curves of order $n$ can be found from any point $P$ and its images under the transformation $x' = x$, $y' = \theta y$, $z' = \theta^n z$, $(\theta^{n^2-n+1} = 1)$. These theorems are applied to space curves, and it appears that the plane curves which are the projections of complete intersections of two surfaces of orders $m$, $m'$ cannot be birationally transformed to curves of order less than $mm' - 1$.

28. In Professor Veblen's note are defined and studied what may be called continuous increasing functions of (finite and transfinite) ordinal numbers. They are such that $x$ and $f(x)$ are both ordinals; if $x_1 < x_2$, then $f(x_1) < f(x_2)$; if $\{x'\}$ is the set of all ordinals less than an ordinal $x$ of the second kind, then $f(x)$ is the least ordinal greater than every $f(x')$. A well-ordered set of these functions is constructed of which Cantor's exponential function $\omega^x$ is the first. The study of this set of functions leads to a classification of Cantor's $\epsilon$-numbers.

29. Lüroth's quartic is characterized by containing the ten intersections of five secant lines; in fact it has an infinite system

* *Nouvelles Annales*, series 2, vol. 9, p. 410.

of inscribed quinquilaterals, whose lines are tangent to a conic. It is associated to a determinate Clebschian quartic which has every such quinquilateral for a polar polygon. Each of these quartics is characterized projectively by a single invariant condition; that of the Clebschian is a known simple invariant, but for the Lürothian the condition has not been worked out. Miss Miller and Professor White show that it can be expressed by equating to zero the resultant of six cubics homogeneous in six unknowns.

This form of quartic is discussed by Lüroth in volumes 1 and 13 of the *Mathematische Annalen;* at the end of his second article he speaks particularly of the present problem. Scorza, in volume 2 (third series) of the *Annali di Matematica*, and in volume 52 of the *Mathematische Annalen*, makes an exhaustive study of the relation between the Lürothian and the Clebschian by the use of the 36 even thetas of three arguments. A short paragraph in Wieleitner's Theorie der ebenen algebraischen Kurven höherer Ordnung treats Lüroth's curve geometrically.

30. In the *Transactions* for January, 1906, and January, 1907, Professor Fite discussed the manner in which the degree of an irreducible linear homogeneous group is determined by the abstract properties of the group. The discussion was limited for the most part to groups whose orders are powers of a prime. In the present paper the same question is considered for a somewhat different category of groups — namely, those in which every non-invariant commutator gives an invariant commutator besides identity. It is shown that the degree of such an irreducible group is uniquely fixed and an expression for the degree is given.

Besides the articles by the author already cited, reference may be made to Frobenius, "Ueber die Primfactoren der Gruppendeterminante," *Berliner Sitzungsberichte*, 1896, II, page 1343, and Burnside, "On the reduction of a group of homogeneous linear substitutions of finite order," *Acta Mathematica*, volume 28, page 369.

31. Given a linear homogeneous substitution of degree $n$ and period $m$, whose coefficients are rational numbers, and such that there is no similar substitution of degree less than $n$, then $n$ is a certain definite number-theoretic function of $m$. This function enables Dr. Ranum to determine all the finite periods of

$n$-ary linear substitutions having rational coefficients. A linear substitution is said to be reducible in a given field if it can be transformed into a reduced form, whose coefficients belong to the field. In the field of rational numbers it is found that every irreducible linear substitution of period $m$ is of degree $\phi(m)$. By means of this and other considerations, all $n$-ary linear substitutions of period $m$ with rational coefficients are easily classified as to their reducibility; finally a method is given of finding a rational canonical form of every such linear substitution.

32. Professor Stephens showed that certain curves, whose equations are of the form

$$t^n - \alpha t^{n-1} + \mu t^{n-2} - \alpha\mu t^2 + \alpha\mu t - \alpha = 0$$

are easily constructed mechanically — the instrument used being similar in principle to that for the pentadeltoid. These curves are of class $n$ and order $2(n-1)$. Each curve is circumscribed to an ellipse, touching it in $n$ points. In any direction there are $n-2$ parallel tangents, the centroid of whose points of tangency is constant. The line at infinity is a double tangent. The cusps are $n$ in number, real or imaginary. The special case $\mu = (n-1)/(n-3)$ exhibits some interesting properties.

33. The possible groups to which algebraic configurations of genus 4 belong have been determined by Wiman,* by means of collineations in space of three dimensions. Miss Van Benschoten completes Wiman's results by interpreting geometrically the various transformations when the configuration is expressed as a plane binodal quintic curve, and by deriving their equations. Every curve $c_5^4$ of order 5 and genus 4 is the plane projection of a space sextic $R_6$ lying on a quadric $F_2$ and a system of cubic surfaces which in particular cases may become cones $K_3$. An illustration is furnished by the octahedron group; it leaves a plane and its pole as to $F_2$ invariant. The six points of $R_6$ in this plane are points of tangency of lines through the pole. $R_3$ lies on 6 $K_3$, whose vertices form a complete quadrilateral in the invariant plane. The points of $R_6$ lie on the sides of the diagonal triangle; they are also in six-fold involution on

† Wiman, "Ueber die algebraischen Kurven von den Geschlechtern 4, 5, 6," Stockholm Academy, *Bihang till Handlingar*, vol. 21 (1895).

the conic containing them. This configuration completely determines the axial and central involutions which leave the curve invariant, and indirectly defines the group. After deriving this and similar results for the other groups, the $R_6$ is projected into $c_2^4$ and the corresponding forms discussed.

34. In the problem of minimizing the integral

$$I = 2\pi \int y \sqrt{x'^2 + y'^2}\, dt$$

between the two fixed points $A_0$ and $A_1$, Miss Sinclair proves that either the catenary or the Goldschmidt discontinuous solution furnishes an absolute minimum with respect to rectifiable curves in the region $y \geqq 0$. She then discusses a discontinuous solution consisting of three curves which meet in a point $A_2$, a catenary through $A_0$ and $A_2$, a second catenary through $A_2$ and $A_1$, and a normal to the $x$-axis. She finds that with respect to similar systems of rectifiable curves in the region $y \geqq 0$ the above extremal system furnishes a relative minimum for the integral under the following necessary and sufficient conditions:

1) The vertices of the two catenaries are equidistant from the $x$-axis, and meet each other and the vertical line through $A_2$ at an angle of 120°.

2) $A_0(t = t_0)$ is limited in position by the inequality $t_0 > t_0^*$, where $t_0^*$ is the unique negative solution of the equation $\coth t - t = 2 + \log \sqrt{3}$, and $A_1(t = t_1')$ is limited by the inequality $t_1 < t_0'$, where $t_0'$ is the unique positive solution of the equation $\coth t - t = \coth t_0 - t_0 - \log 3$, and $A_0'(t = t_0')$ is the conjugate of $A_0$, and is the point in which the system touches the envelope of the set of extremal systems through $A_0$ which satisfy condition 1). $A_0'$ may also be located by a Lindelöf construction. The value of $I$ furnished by this discontinuous solution is for a certain region less than that given by the Goldschmidt solution, but never less than that given by the single catenary through $A_0$ and $A_1$. The system may be constructed by means of liquid films. Actual measurements of the limit of stability of the film correspond nicely to the theoretical values for the conjugate point.

35. The first aim of Dr. Fréchet's paper is to generalize the known results concerning linear functional operations (and particularly Hadamard's theorem) to cover the case where these

operations are defined not only, as before, in the field of continuous functions (of a single variable) but also in the far greater field of the functions which have an integral according to Lebesgue's definition. The second point is to prove that any of the operations defined in such a field can be determined by one continuous function $u(x)$ of a single variable, called characteristic function of the operation. The operation $U_f$ is numerically known by means of the limit of one double integral when $f(x)$ and $u(x)$ are known. Conversely $u(x)$ is obtained by applying the operation to the function $\phi_y(x)$, $\phi_y(x)$ being equal to 1 when $0 \leqq x \leqq y$ and to 0 when $y < x \leqq 2\pi$.

36. The original object of Professor Greenhill's memoir was to provide the simplest method of calculating numerically the electromagnetic constant of a helix, in the ampère balance, designed by the late Viriamu Jones and Professor Ayrton for weighing the electromagnetic attraction and so arriving at an independent measure of the electrical units.

Incidentally it was requisite to coordinate the conflicting notation of various writers on the subject, by adopting Maxwell's Electricity and Magnetism as a standard, and to develop Landen's quadric transformation in order to reconcile results apparently discordant, so that the fundamental geometric meaning of Landen's transformation appears in the course of the electromagnetic theory.

The analytic complexity in the reduction of the elliptic integral in electromagnetism, as well as in most dynamical problems arises in consequence of the appropriate integral of the third kind being of the circular form in Legendre's terminology, the elliptic parameter of Jacobi is then a fraction of the imaginary period, so that the expression by the theta function implies a complex argument and a table of the theta function would not be of complete utility unless made in a triple entry form.

But in the practical problems of electromagnetism it is the complete third elliptic integral which is sufficient for a solution, and this, as Legendre has shown, can be expressed by integrals of the first and second kind, complete and incomplete, for which Legendre's Table IX provides the numerical value.

In working with the third elliptic integral it is found a practical convenience to retain it in the algebraical form, and to delay the adoption of a Legendrian, Jacobian, or Weierstrassian notation; the circular form of the complete integral will then

fall into one of four classes, according to the sequence of the quantities involved.

The geometric interpretation of a formula has been explained wherever possible, when used for the expression of an electromagnetic quantity, as mutual induction, magnetic potential, vector potential, and so forth; and the advantage of the Stokes function in analytic simplicity has been emphasized over the ordinary potential function.

F. N. COLE,
*Secretary.*

---

## ON A SPECIAL ALGEBRAIC CURVE HAVING A NET OF MINIMUM ADJOINT CURVES.

BY PROFESSOR VIRGIL SNYDER.

(Read before the American Mathematical Society, September 5, 1907.)

IN researches on plane curves a very fruitful configuration has been employed by Küpper* in obtaining particular curves having double points in abnormal position. His method consists in using as basis points part of the intersections of two curves which satisfy certain prescribed conditions; pencils and nets are then constructed having these basis points as nodes. The question naturally arises whether the same procedure can be employed in other cases, making use of all the constants in the system. In the following note it will be shown that in such cases the special series which are obtained cannot be employed to reduce the order of the curve. Incidentally, an illustration is furnished of configurations having a $g^2_{2(n-1)}$ (a linear series with two degrees of freedom and of order $2(n-1)$), although the curve cannot be reduced to order $2(n-1)$.

1. Through $n-1$ points on a straight line pass two general curves of order $n$, $c_n$, $c'_n$. These curves intersect in $n^2-n+1$ residual points through which can be passed $\infty^2$ curves of order $n$. Any two points $P$, $Q$ of the plane will determine two pencils $c_n+\lambda c'_n=0$, $\phi_n+\lambda\phi'_n=0$ contained in this net, which can so be made projective that the locus of the variable intersections is the curve $c_{2n}$ of order $2n$

$$c_n\phi'_n-c'_n\phi_n=0. \tag{1}$$

*C. Küpper: "Ueber das Vorkommen von linearen Schaaren . . . ," *Sitzungsberichte der Böhmischen Gesellschaft*, Prag, 1892, pp. 264–272.

The $n^2 - n + 1$ basis points are all double points of $c_{2n}$; the net $c_n$ defines upon it a $g^2_{2(n-1)}$. If $c_{2n}$ has no other double points, the net defines an adjoint system. The canonical adjoint curves of order $2n - 3$ can be composed of $\phi_n$ and any fixed curve of order $n - 3$. The usual form of the theorem of birational transformation is that any system of adjoint curves of order $2n - 3$ and having two degrees of freedom may be used for transforming curves, but for our net the double points furnish the minimum number of conditions for a curve of order $n$. Moreover, the $g^2_{2(n-1)}$ is of such a nature that one point determines $n - 2$ others, analogous to the case of the hyperelliptic curves. If $n > 3$, $p > 2n - 2$, hence $c_{2n}$ cannot be hyperelliptic.

2. When $n = 3$, the $g^2_4$ is composed of two $g^1_2$, the groups being associated arbitrarily, hence it is hyperelliptic.

*The necessary and sufficient condition that a sextic curve with seven double points be hyperelliptic is that every pencil of cubic curves through the seven double points and one simple point on the curve shall have the residual basis point on the curve also.*

The curve can be reduced to a quintic with a triple point by means of adjoint quartics; it cannot be reduced to a quartic curve.

3. If we now write $\rho x_1 = c_n \chi_{n-3}$, $\rho x_2 = c'_n \chi_{n-3}$, $\rho x_3 = c''_n \chi_{n-3}$ then (1) will go into a conic section, but it will be shown that (1) can never be rational. If we use (for a moment) the notation of Amodeo in his memoir on minimum adjoint curves (*Rendiconti dei Lincei*, series 5, volume 2 (1893), pages 460–467) we have $m = 2n$, $p = n^2 - 2n$, $\alpha = b - 3$, $\rho = \frac{1}{2}(n - 2)(n - 3)$, wherein adjoint curves of $c_m$ of order $m - 3 - \alpha$ exist and will pass through the remaining $\rho$ double points if required to pass through all but $\rho$. As is shown by Amodeo, if these conditions are satisfied, then the intersection of $c_{2n}$ with any $c_{n-3}$ (not adjoint), forming a $g^{n/2(n-3)}_{2n(n-3)}$, will not be a complete series, but will be contained in another having $\rho$ further degrees of freedom. Equation (1), page 466, is satisfied if $t = 1$, hence all the formulas are satisfied when our $c_{2n}$ has the maximum genus $n^2 - 2n$.

4. Any two curves of the net will intersect in $n - 1$ points lying on a straight line. The pencil of adjoint curves which has any fixed basis point on $c_{2n}$ will have $n - 2$ other fixed basis points upon it, and define a $g^1_{n-1}$. The $n - 1$ points of each group of this series are collinear. Such a series can be defined by starting from any point of $c_{2n}$ as basis point, but from

the form of the equation it immediately follows that *all the* $g^1_{n-1}$ *are identical.* The groups of $g^1_{n-1}$ determine an $[n-2]$ involution on $c_{2n}$. If $p$ be the genus of $c_{2n}$, the number of coincidences is $2(n+p-2)$.*

If $c_{2n}$ has a double point apart from the $n^2-n+1$ basis points, it will count for two coincidences, hence if $c$ be the number of simple coincidences we have

$$2(n+p-2) = 2(n^2-2n-p)+c$$

or

$$4p = 2n(n-3)+c+4.$$

Since $c$ cannot be negative, the minimum value of $p$ is that of the general $c_n$, thus

$$\tfrac{1}{2}(n-1)(n-2) \leqq p \leqq n(n-2).$$

If singular $G_{n-1}$ exist in $g^1_{n-1}$, double points may arise which do not involve two coincidences, but they can only occur for particular relations among the coefficients of $c_{2n}$, which we exclude.

Since $p>0$, we can now state the following results:

(*a*) *The complete series* $g^2_{2(n-1)}$ *cannot be used for birational transformation.*

(*b*) *The envelope of the straight lines on which lie the groups of* $g^1_{n-1}$ *is a conic.*†

(*c*) *The normal or canonical form of* $c_{2n}$ *is a conic, taken* $(n-1)$ *fold.*

The analogy between these curves and the hyperelliptic curves thus becomes apparent.

5. When $n=4$, it was seen that the straight lines of the plane cut from $c_8$ a partial series $g^2_8$ contained in the complete series $g^3_8$, hence $c_8$ is the projection of a space curve of the same order. Curiously, the same thing is true for every value of $n$. Consider the space curve on the quadric surface, cutting the generators of one system in $n+1$ points, and those of the other in $n-1$ points. The space curve $R_{2n}$ has therefore $h=n^2-n+1$

* From the Cayley-Brill principle of correspondence, or from Zeuthen's formula. See Bertini: "La geometria delle serie lineari sopra una curva piana secondo il metodo algebrica," *Annali di Mat.* (2), vol. 22 (1894), pp. 1–40.

† Amodeo, "Curve $\kappa$-gonali," *Ann. di Mat.* (2), vol. 21 (1893), pp. 221–336. See formula, bottom page 231. $m=2n$, $\kappa=n-1$, $t=1$, hence $y$, the class of the envelope, is 2.

apparent double points. If now we project into $c_{2n}$ from a center not on the quadric, the $n^2 - n + 1$ double points will be distinct. We thus have the theorem: "If through $h - \frac{1}{2}i(i+1)$ of the $h$ double points of the projections curve $c_{2n}$ a curve of order $2n-i-3$ can be passed $(2n-i-3 \geqq n-1)$, then this curve will pass through the remaining $\frac{1}{2}i(i+1)$ double points also."*

Through $\frac{1}{2}n(n+3) - 2$ arbitrary points can be passed $\infty^2$ $c_n$. If $i = n-3$, we see from the above theorem that all such curves $c_n$ through $\frac{1}{2}n(n+3) - 2$ of the double points will pass through the others also, hence a net of adjoint curves of order $n$ exists for our projection curve $c_{2n}$. The adjoint curves define upon $c_{2n}$ a $g^2_{2(n-1)}$. In case $n = 4$, the $g^2_6$ is the residual of $g^3_8$ by the Riemann-Roch theorem, but in no case can this net be used to transform the curve $c_{2n}$ birationally into another of order $2(n-1)$. If we project $R_{2n}$ from a point of the quadric on which it lies, $c_{2n}$ will have an $(n-1)$ fold point $P_{n-1}$ and a $P_{n+1}$, and in general no other singularities. The $g^1_{n-1}$ defined by one system of generators is thus defined by one pencil of lines, and the $g^1_{n+1}$ by another. By projecting $R_{2n}$ from a point upon the curve, or by ordinary inversion of $c_{2n}$ we obtain a $c_{2n-1}$ with a $P_n$ and a $P_{n-2}$, and no other singularities. By inversion, an infinite number of projectively distinct $c_{2n-1}$ can be obtained, but they will all have multiple points of the same order. In no case can this curve be reduced to a curve of lower order by any birational transformation.†

When $n = 4$, $c_8$ has a $g^1_3$ which is invariant, but when $c_{2n}$ has a $g^1_3$, $p \leqq 2m - 5$; since $c_8$ is of genus 8, it follows that $m \geqq 7$.‡

Probably similar deductions can be made for larger values of $n$ from the memoirs of Amodeo and of Bertini. Any adjoint $\phi_n$ containing a point $P$ must also contain the whole group $G_{n-1}$ to which $P$ belongs (Bertini, l. c., page 29), hence if two different $\phi_n$ pass through $P$, the $n-1$ points of intersection will be the $G_{n-1}$ to which $P$ belongs. These points always lie on a straight line (Bertini, page 30, *b*). This shows that our projection curve $c_{2n}$ can be generated projectively by the method of § 1, and the

* H. Valentiner, "Zur Theorie der Raumkurven," *Acta mat.*, vol. 2 (1883), pp. 136–230. See p. 177. Also M. Noether, "Zur Grundlegung der Theorie der algebraischen Raumkurven," *Berliner Abhandlungen* (1883), pp. 1–120. See p. 24.

† V. Snyder. "On birational transformation of curves of high genus," *Amer. Jour. Math.*, vol. 30 (1908), pp. 11–20.

‡ K. Bobek: "Ueber Dreischaarkurven," *Wiener Berichte*, vol. 98 (1889), pp. 141–172.

two systems are thus coextensive. Since on $R_{2n}$ the $g^1_{n-1}$ is defined by the generators of one system of the hyperboloid, *the envelope of the lines containing the $G_{n-1}$ is the section of the tangent cone to the hyperboloid from the center of projection.*

6. If $\phi_n$, $\phi'_n$, $\phi''_n$ be three cones of order $n$ containing the $n$ bisecants from (0, 0, 0, 1), then, since the $n-1$ remaining edges of intersection with the cone $R_{2n}$ from the same point lie in a plane, the equation of the defining monoid may be written

$$w = \frac{x\phi'_n}{\phi_n} = \frac{x\phi''_n}{\phi'_n},$$

from which the equation (1) results. Incidentally, these equations furnish a means for reducing $c_{2n}$ to $c_{2n-1}$, namely, the $\infty^2$ plane sections of the monoid from a point on $R_{2n}$, lying on one of the $n-1$ simple edges.

7. If $R_{2n}$ has also actual double points or cusps, $\phi_n$ will not in general pass through them, hence in the plane curves $c_{2n}$ we can distinguish between projection of actual double points and apparent double points. Actual double points will not always absorb two coincidences in the $[n-2]$ involutions, but when $p < \frac{1}{2}(n-1)(n-2)$, the projection curve can not be written in the form (1).

For other curves on the hyperboloid, the maximum number of basis lines of a net formed by bisecants will not be reached; but when no actual double points occur we may say that the projection curve $c_n$ with $p > \frac{1}{6}(n-1)(n-2)$ cannot be birationally transformed into any curve of order less than $n-1$.

CORNELL UNIVERSITY,
*August*, 1907.

---

## NOTE ON CERTAIN INVERSE PROBLEMS IN THE SIMPLEX THEORY OF NUMBERS.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society, September 5, 1907.)

Legendre* has considered the problem of finding the highest power of a prime $p$ contained in $m! = 1 \cdot 2 \cdot 3 \cdots m$. Let $m$ be written in the form

$$(1) \qquad m = a_0 p^\alpha + a_1 p^\beta + a_2 p^\gamma + \cdots,$$

* A. M. Legendre, Théorie des nombres, 3d ed., I., p. 10.

where $\alpha, \beta, \gamma, \cdots$ are different positive integers (including zero) and $a_0, a_1, a_2, \cdots$ are the same or different positive integers each less than $p$. Put

$$a_0 + a_1 + a_2 + \cdots = s. \tag{2}$$

Then the highest power of $p$ contained in $m!$ is $p^{(m-s)/(p-1)}$; or, in the congruence

$$m! \equiv 0 \pmod{p^{(m-s)/(p-1)}}, \tag{3}$$

the modulus gives the highest power of $p$ for which the congruence holds.

The problem first to be considered in this note is to find the solutions of (3) when $p$ and $s$ are known and $m$ is the unknown.*

Suppose a solution

$$m_1 = a_0 p^\alpha + a_1 p^\beta + a_2 p^\gamma + \cdots \tag{4}$$

has been found. Take $\alpha$ to be the least of the numbers $\alpha, \beta, \gamma, \cdots$ and divide equation (4) by $p^\alpha$. We obtain

$$\frac{m_1}{p^\alpha} = a_0 + a_1 p^{\beta-\alpha} + a_2 p^{\gamma-\alpha} + \cdots = m_2 \text{ say}. \tag{5}$$

It is clear from the first paragraph that $m_2$ is also a solution of (3). Moreover if $m_2$ is multiplied by any power of $p$ the resulting number is also a solution of (3), subject to all the conditions which have been imposed. Therefore,

*To every solution of* (3) *there corresponds an infinite number of solutions differing only by factors which are powers of* $p$.

We shall define a characteristic solution to be such a one as that given in (5); thus, *a characteristic solution of* (3) *is one for which* $m$ *is not divisible by* $p$.

Now suppose $s$ separated in any way into parts each of which is less than $p$ and let these parts be $a_0, a_1, a_2, \cdots$. Then (1) is a solution of (3) however $\alpha, \beta, \gamma, \cdots$ may be chosen so that

* The modified problem of finding the solutions of the congruence

$$m! \equiv 0 (\mathrm{mod}\ p^v),$$

subject to the limitation that $p^v$ is the highest power of $p$ in $m!$, is very easy. It is clear that the smallest solution is always a multiple of $p$, say $= \mu p$. Then other solutions are $\mu p + 1, \mu p + 2, \cdots, \mu p + p - 1$; moreover these are all the possible solutions. It is to be noticed that a solution of this problem does not always exist. A case in point is $p = 3$, $v = 3$; for these values there is no solution of the congruence subject to the imposed conditions.

they are different positive integers including zero. Hence, when $s > 1$ so that this partition is possible, *there is an infinite number of solutions for each partition of $s$ as above.* When $s = 1$ it is clear that the solutions are $m = 1, p, p^2, \cdots$. Furthermore it is evident that there can be no solutions except those which are here obtained by aid of the partitions of $s$ into parts each less than $p$.

As a second problem we shall consider the solution of

$$m! \equiv 0 \pmod{p^{m-t}}, \tag{6}$$

where $p^{m-t}$ is the highest power of $p$ contained in $m!$, $m$ still being the unknown. For every solution $m$ we have from the proposition of the first paragraph

$$m - t = \frac{m - s}{p - 1}; \tag{7}$$

whence, if $p \neq 2$, that is, if $p$ is an odd prime,*

$$m = \left(1 + \frac{1}{p-2}\right)t - \frac{s}{p-2}, \tag{8}$$

where $s$ as before equals $a_0 + a_1 + a_2 + \cdots$, the sum of the coefficients of the powers of $p$ when $m$ is expressed in the form of (1). Therefore, since

$$m < \left(1 + \frac{1}{p-2}\right)t, \tag{9}$$

a definite constant, *the number of solutions of this problem is finite.*

Now suppose

$$m = c_n p^n + c_{n-1} p^{n-1} + \cdots + c_1 p + c_0.$$

Then

$$s = c_n + c_{n-1} + \cdots + c_1 + c_0.$$

Hence from (8) we may write

$$\begin{aligned} &c_n p^n + c_{n-1} p^{n-1} + \cdots + c_1 p + c_0 + \frac{c_n + \cdots + c_1 + c_0}{p-2} \\ &= \left(1 + \frac{1}{p-2}\right)t = r_\nu p^\nu + \cdots + r_1 p + r_0 + \frac{u}{p-2} \end{aligned} \tag{10}$$

* If $p = 2$, $t = s$; this problem therefore reduces to the preceding for the case $p = 2$. Its solution has already been found. We shall henceforth consider $p$ an odd prime.

where $m$ and $s$ are replaced by their values above and $r_\nu, \cdots, r_0$ are integers each less than $p$ and $u < p-2$. The $r$'s are uniquely determined from the known value of $[1+1/(p-2)]t$. Now

$$(11) \qquad \frac{c_n + \cdots + c_0}{p-2} \equiv \frac{(n+1)(p-1)}{p-2} < p^n, \text{ if } n > 1.$$

Therefore in equation (10) $n = \nu$, when $n > 1$; then also $c_n = r_\nu$. Having obtained these values we may proceed similarly to consider the coefficeints $c_{n-1}$, $r_{\nu-1}$ and so on. All the possible solutions will be obtained with but little difficulty. The work will be made easy from the fact that the coefficients of like powers of $p$ are the same in the two members of (10) except for small exponents of $p$.

As an example, consider the solution of the problem when $p = 11$ and $t = 14578$. We have

$$\left(1 + \frac{1}{p-2}\right)t = 16197\tfrac{7}{9} = 11^4 + 11^3 + 11^2 + 9 \cdot 11 + 5 + \tfrac{7}{9}.$$

Comparing with (10) we have $n = 4$, $c_4 = 1$. Then without difficulty we may further show that $c_3 = 1$ and $c_2 = 1$. Substituting in (10) and reducing we have

$$c_1 \cdot 11 + c_0 + \frac{c_1 + c_0}{9} = 9 \cdot 11 + 5 + \tfrac{4}{9}.$$

From this equation it is readily shown that $c_1 > 8$ and $< 10$; and therefore its only possible value is $c_1 = 9$. This gives $c_0 = 4$. Therefore,

$$m = 11^4 + 11^3 + 11^2 + 9 \cdot 11 + 4 = 16196.$$

There is nothing in the preceding discussion to show that a solution always exists for this problem. An examination of the product of the natural numbers in order, reference being had to how often any prime has entered up to any number $m$, will easily lead to the discovery that the consecutive values of $t$ contain every integer beginning with 1 and that some integers will be repeated in the series of values of $t$. Hence, *there always exists at least one solution; and in some cases there is more than one solution.*

ANNISTON, ALA.,
*May*, 1907.

# THIRD REPORT ON RECENT PROGRESS IN THE THEORY OF GROUPS OF FINITE ORDER.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, September 6, 1907.)

THE following general treatises have appeared since the presentation of my second report * a little more than five years ago: The constructive development of group theory with a bibliography, by B. S. Easton, 1902; A brief work in Russian entitled The polyhedron groups, by G. W. Pfeiffer, 1903; Mathematical crystallography and the theory of groups of movements, by H. Hilton, 1903; Eléments de la théorie des groupes abstraits, par J. A. de Séguier, 1904; and Quelques considérations sur les groupes d'ordre fini et les groupes finis continus, par Raymond Le Vavasseur, 1904. The number of treatises devoting at least one chapter to our subject has been considerably increased. Two have appeared in America: Dickson, Introduction to the theory of algebraic equations, 1903, and Cajori, An introduction to the modern theory of equations, 1904.

Unfortunately the report by Dr. Steinitz, which was announced by the Deutsche Mathematiker-Vereinigung at different times had to be abandoned. The works by Loewy and Wiman entitled respectively Vorlesungen über die Theorie der linearen Substitutionsgruppen and Endliche Gruppen linearer Transformationen, which were expected to appear in Teubner's Sammlung, have also been delayed. On the other hand Netto is preparing an elementary work for the Sammlung Schubert under the title Gruppen und Substitutionentheorie, and the second part of de Séguier's work is in press.

In view of the very large number of new theorems it appears impracticable to attempt to enumerate all of them. In the case of the larger memoirs where a number of connected theorems are developed I can generally only give such references as indicate the nature of the main questions, while the theorems which have appeared in very brief articles or in less accessible places receive relatively more attention. Moreover, it did not appear best to crowd the report with a complete list

* BULLETIN, November, 1902, p. 106.

of references, since bibliographical works are more convenient for the purpose of learning all the details along special lines. I hope to have succeeded in exhibiting the connections between some isolated developments which are of importance, and that the report may thus arouse a wider interest along certain lines. It may be added that the advances in the theory of abstract groups have been noted in greater detail than those which come under the other headings of this report.

## § 1. Elementary Applications.

Among the most elementary and the most important applications of groups of finite order are those which relate to the fundamental operations of arithmetic. The addition and the multiplication groups * were developed principally during the periods covered by the preceding reports. One of the most important extensions of the theory of the latter, made during the period covered by the present report, is given by the following theorem: The necessary and sufficient condition that a given set of numbers, which are distinct with respect to modulus $m$, generates a group with respect to multiplication is that the highest common factor of the modulus and a number of the set is independent of the choice of this number and that the quotient obtained by dividing $m$ by this highest common factor is prime to this factor.†

If $x_1$ and $n$ represent any two numbers (real or complex), the number $x_1 - n$ can be obtained geometrically by reflecting the point representing $n$, in the complex number plane, on the middle point of the segment connecting $x_1$ with the origin; that is, by connecting the point representing $n$ with this middle point and extending the segment to an equal distance beyond this point. From this standpoint subtraction may clearly be regarded as an operation of period 2; and, if $n$ is subtracted from a series of numbers $x_1, x_2, \cdots, x_l$ and the remainders are again subtracted from these numbers, in any order, these operations of period 2 generate a dihedral group and every dihedral group may be generated in this way. It is of interest to observe that the series of numbers $x_1, x_2, \cdots, x_l$ may be replaced by two numbers without affecting this dihedral group,‡ but a further

* Bulletin, vol. 7 (1900), p. 122; *Annals of Mathematics*, vol. 2 (1901), p. 77.

† *Annals of Mathematics*, vol. 6 (1905), p. 44.

‡ Loc. cit., p. 43.

reduction is generally impossible. As the group is dependent merely upon the difference of these numbers and the modulus it is clear that every dihedral group may be generated in an infinite number of ways as a subtraction group.

The applications of these groups which have been most fully developed are those connected with the notions of complement and supplement of an angle in elementary trigonometry. As the modulus in this case is 360°, this group is equivalent to the one obtained by subtracting from 1 and 2 with respect to modulus 4. Hence it is identical with the group of movements of the square. The fact that there are two sets of four angles which do not go into eight distinct angles by means of the operations of taking the complement and the supplement is thus related to the general theory that, with respect to the subtraction group, there are always two and only two such sets of numbers, and the total number of different numbers in the two sets is equal to the order of the dihedral group. The more general applications of the subtraction groups in elementary trigonometry have also been considered and a few illustrative examples are given in a recent article in the *Annals of Mathematics.**

Results of unusual interest relate to the groups generated by the two operations of subtracting from a fixed number and dividing either the same or a different number. These groups differ from those considered above with respect to the fundamental property that it is not necessary to assume a modulus in order to obtain finite orders. If the subtrahend is denoted by $x_1$ and the dividend by $x_2$, the two operations $x_1 - n, x_2/n$ generate a dihedral group, since each of these operations is of order 2. The important result that only four distinct groups of finite order are possible when $x_1$ and $x_2$ are both rational numbers has been proved both analytically and geometrically.† The orders of these groups are 4, 6, 8, and 12. These orders are obtained by assuming $x_1 = 0$, $x_1^2 = x_2$, $x_1^2 = 2x_2$, and $x_1^2 = 3x_2$ respectively. There are three sets of conjugate numbers under such a group which include all the numbers which are transformed into a smaller number of distinct values than the order of the group.‡

---

* Vol. 8 (1907), p. 97.

† *Quar. Jour. of Mathematics*, vol. 37 (1905), p. 80; Hilton, *Messenger of Mathematics*, vol. 35 (1905), p. 117.

‡ *Quar. Jour. of Mathematics*, loc. cit.

If it is not assumed that $x_1$, $x_2$ are rational, it is possible to assign them values so that the two operations $x_1 - n$, $x_2/n$ generate a dihedral group of any given possible order. One set of values which leads to the dihedral group of order 10 is $x_1 = 1 + \sqrt{5}$, $x_2 = 4$. Among the sets which give rise to the symmetric group of order 6, that in which $x_1 = x_2 = 1$ has received most attention, as this is the group formed by the six values of an anharmonic ratio as well as by the six trigonometric functions $\sin^2\alpha$, $\sec^2\alpha$, $-\cot^2\alpha$, $-\tan^2\alpha$, $\csc^2\alpha$, $\cos^2\alpha$. As the four-group is the only invariant subgroup (besides identity) of a symmetric group which gives rise to a quotient group whose order exceeds 2, the six values of an anharmonic ratio furnish a unique instance of a more than two valued rational function all of whose values are rationally expressible in terms of one of them. That is, any other function having this property is rationally expressible in terms of an anharmonic ratio.

Another instance of very elementary applications which were developed during the period covered by the present report is furnished by the groups of the figures of elementary geometry.* The group of a figure is composed of all the movements of space which transform the figure into itself. In these movements space is regarded as rigid in the sense that any two points are transformed into two points at the same distance from each other. Every movement which transforms a system of points into itself must transform the center of their minimum circumscribing sphere into itself. Hence such movements must be composed of rotations around this center, and all the groups of movements of a system of coplanar points, finite in number, must be either cyclic or of the dihedral rotation type. Moreover, the dihedral group of order $2n$ is the group of the regular plane polygon of $n$ sides. In particular, the octic group is the group of movements of the square; while the rectangle with unequal sides has the transitive four-group for its group of movements, and the rhombus which is not a square has the intransitive four-group for its group of movements. If the base of a regular pyramid has $n > 3$ sides, its group is cyclic and of order $n$, and if the base of a regular prism has $n$ sides ($n \neq 4$), its group is dihedral rotation and of order $2n$.

During the last few years some of the direct applications of

* *Amer. Math. Monthly*, vol. 10 (1903), p. 215.

substitution groups in elementary trigonometry have received notice.* If $a$, $b$, $c$ represent the sides of a spherical triangle, and $\alpha$, $\beta$, $\gamma$ the supplements of the angles opposite those sides in order, it follows from the properties of the polar triangle that the substitution $a\alpha \cdot b\beta \cdot c\gamma$ transforms any formula relating to the spherical triangle into one which is equally true. It is also evident that any formula which has been derived without specializing any one of the six parts in question is transformed by the two substitutions $ab \cdot \alpha\beta$, $bc \cdot \beta\gamma$ into one which is equally true. Hence it follows that the group of order 12 which is generated by these three substitutions transforms any such formula into those which are equally true. This well known imprimitive group of order 12 and degree 6 may therefore be employed to derive a system of formulas from a given one, and hence it may sometimes be used to advantage even in a very elementary course of instruction. Its intransitive subgroup of order 6 may be employed to some advantage in plane trigonometry.

## § 2. Abstract Groups.

During the period under consideration great progress was made in the theory of abstract groups, just as had been the case during the preceding period. I will first state some of the new theorems relating to generational operators. If a series of operators is such that each of them transforms all the others either into their inverses or into their third powers, the totality of these operators generates the hamiltonian group of order $2^n$, except when all of them are of order 2.† In this special case it generates the abelian group of order $2^m$ and of type $(1, 1, 1, \cdots)$. If a non-abelian group is generated by two operators which transform each other into their $\alpha$, $\beta$ powers respectively, the group is completely defined by these powers whenever the indices of the powers diminished by one are such that one of them is a prime while the other is a power of this prime, and only then. In other words, the non-abelian group of order $p^m$ which contains a cyclic subgroup of order

* *Quar. Jour. of Mathematics*, vol. 37 (1906), p. 226.

† From this theorem we may readily derive two very simple definitions of the quaternion group, viz., the quaternion group is the only non-abelian group generated by two operators which transform each other into their third powers, and the quaternion group is the only non-abelian group generated by two operators which transform each other into their inverses. *Prace Matematyczno-fizyczne*, vol. 17 (1906), p. 120.

$p^{m-1}$ is the only group, besides the quaternion, which is completely defined by the facts that it is non-abelian and that it is generated by two operators which transform each other into given powers. Any series of operators such that each is transformed into a power of itself by all the others generates an abelian or a metabelian group which is therefore the direct product of its Sylow subgroups, but not every metabelian group can be generated in this way.*

If three operators are such that the product of any two is equal to the third, they generate either a dihedral group or the quaternion group.† If $n$ different operators are commutative and if each of them is the product of all the others, they generate the direct product of a cyclic group whose order divides $2(n-2)$ and an abelian group of order $2^n$ and of type $(1, 1, 1, \cdots)$. Moreover, any such direct product may be generated by $n$ operators which satisfy the given conditions. If the order of one of these operators is divisible by 4, all of them are of the same order. Every possible symmetric group whose degree exceeds 3 may be generated by five non-commutative operators such that each of them is the continued product of the other four, arranged in a given order. The groups generated by four operators such that each is the product of the other three, arranged in a given order, are similar to the dihedral type and have been completely determined.‡

It is well known that the groups of genus zero exhibit the most interesting generational relations. All of these have recently been generalized by replacing two of the three ordinary equations which define such a group by a single one while the third is maintained in the usual form. A large number of new and comparatively simple definitions of groups are thus obtained.§ Closely related to the groups of genus zero are those which may be defined by the orders of two operators and the order of their commutator. When two operators of order 2 have a commutator of an even order, they generate a group whose order is four times the order of this commutator; when they have a commutator of odd order, they generate a group whose order is either twice or four times the order of their commutator. If an operator of order 2 and an operator of order 3

* *Quar. Jour. of Mathematics*, vol. 37 (1906), p. 286.
† BULLETIN, vol. 13 (1907), p. 382.
‡ BULLETIN, vol. 13 (1907), p. 433.
§ *Transactions Amer. Math. Society*, vol. 8 (1907), p. 1.

have a commutator of order 2, they generate either the tetrahedral group or the direct product of this group and the group of order 2. If two operators of order 3 have a commutator of order 2, they generate a group whose order is one of the four numbers 12, 36, 144, 288.*

Netto recently published a very elementary article on the construction of abstract groups which are generated by two operators,† in which he obtains a number of well-known results by the use of the most elementary principles and also arrives at a few interesting new relations. It is to be regretted that he failed to indicate where his work has contact with older publications. For instance, the conditions $a^3 = 1$, $b^3 = 1$, $ba = a^2b^2$ on page 255 of Netto's article are clearly equivalent to the well known abstract definition of the tetrahedral group, since $(ba)^2 = 1$, $ba$ being equal to its inverse; yet nothing whatever is stated about this and the reader who is not familiar with the subject would naturally infer that the results were due to Netto. The last section of the article is devoted to the proof that the group generated by two operators of order 4 whose product is of order 2 has an infinite number of distinct operators. This is a very special case of an older general theorem in regard to the groups that may be generated by two operators when only their orders and the order of their product are given.‡

In volume 9 of the BULLETIN, Dickson gave useful generational relations for the simple groups of orders 360, 504, and 660, and in volume 35 of the *Proceedings of the London Mathematical Society* he gave such relations for an abstract simple group of order 4080 as well as for the more general abstract group which is simply isomorphic with the linear fractional group in $GF[2^n]$. In a later volume of the latter journal Bussey gave the generational relations for an abstract group simply isomorphic with $LF[2p^n]$.§ Le Vavasseur, Potron, and Neikirk have given a very large number of generational relations as regards the groups of order $p^m$.‖

One of the most fundamental recent theorems relating to abstract groups is due to Frobenius. It affirms that the num-

* BULLETIN, vol. 13 (1907), p. 371. Cf. *Proc. Amer. Philosophical Society*, vol. 46 (1907), p. 146.

† Netto, *Crelle*, 128 (1905), p. 243.

‡ *Amer. Jour. of Mathematics*, vol. 24 (1902), p. 96.

§ Bussey, *Proc. London Math. Society*, vol. 3 (1905), p. 296.

‖ Le Vavasseur, loc. cit., p. 26; Potron, Thèse, Paris, 1904; Neikirk, thesis, *Publications of the University of Pennsylvania.*

ber of the operators whose $n$th power is equal to a given operator $A$ is divisible by the highest common factor of $n$ and $g$, where $g$ represents the number of the operators of the group which are commutative with $A$.* Another recent fundamental theorem relates to the form of the total number of cyclic groups of order $p^m$ $(m > 1)$ and asserts that the number of cyclic subgroups of order $p^m$ in any group $G$ is always of the form $kp$ whenever the Sylow subgroups of order $p^\alpha$ in $G$ are non-cyclic, and that the number of subgroups of order $p$ in such a $G$ is of the form $1 + p + kp^2$, $p$ being an odd prime number.† If a group of order $2^m$ contains an odd number of cyclic subgroups of order $2^\alpha$ $(\alpha > 2)$ this number must be unity and the group of order $2^m$ contains a cyclic subgroup of order $2^{m-1}$. Moreover the number of operators of order 2 in any group of order $2^m$ which does not involve an operator of order $2^{m-1}$ is always of the form $4k + 3$, $k$ being a positive integer. If a group of order $2^m$ contains exactly two cyclic subgroups of order $2^{m-1}$, the number of the operators of order 2 is also of the form $4k + 3$; but when it contains only one such cyclic subgroup, this number is of the form $4k + 1$.‡ If a Sylow subgroup of order $p^\alpha$ in $G$ is cyclic, the number of the subgroups of order $p^m$ is evidently of the form $1 + kp$.

The study of the groups whose subgroups possess given properties has led to interesting results. From the properties of the non-abelian groups in which every subgroup is abelian, Dickson deduced the form of the order of groups which are necessarily abelian.§ This result is closely related to properties of the orders of groups which are necessarily cyclic. These properties were stated in my first report, page 235. The results as regards the non-abelian groups in which every subgroup is abelian have been extended by the determination of all the non-abelian groups in which every subgroup is either abelian or hamiltonian, as well as all those in which every subgroup is either abelian or dihedral.‖ These conditions led to remarkable elementary categories of groups which will probably continue to be useful, in view of their simple definitions.

All of the groups which were considered in the preceding

* Frobenius, *Berliner Sitzungsberichte*, 1903, p. 987.

† *Proc. London Math. Society*, vol. 2 (1904), p. 142.

‡ *Transactions Amer. Math. Society*, vol. 6 (1905), p. 58.

§ Dickson, *Transactions Amer. Math. Society*, vol. 6 (1905), p. 201.

‖ *Transactions Amer. Math. Society*, vol. 8 (1907), p. 1; *Amer. Jour. of Mathematics*, vol. 29 (1907), p. 289.

paragraph are solvable. This is also true of the category which satisfy the condition that any two conjugate operators are commutative, as well as of the historically more interesting category composed of the groups whose orders are of the form $p^\alpha q^\beta$, $p$ and $q$ being prime numbers.* Special cases of such groups had been proved solvable by Frobenius, Jordan, Cole, and others before Burnside succeeded in establishing this property for the entire category. This seems the most interesting result as regards solvability which was reached during the period under consideration.

Bachmann has observed that the theory of congruences and Fermat's theorem are based upon the "fundamental mathematical concept of groups" and he employed the elements of this subject in his classic work on number theory. A large number of relations between number theory and group theory have recently been considered.† In many instances the known theorems of these subjects express the same fundamental facts in different languages. This is the case as regards the generalized Wilson's theorem and the theorem giving the continued product of all the operators of an abelian group,‡ which was first stated by Zsigmondy. As Zsigmondy's definition of a group is not in accord with later usage, some of the results cannot be directly employed and hence his valuable memoir has not received the notice which it deserves. For instance, it gives a correct statement of the possible types of subgroups of abelian groups,§ while Burnside in his later Theory of Groups came to erroneous conclusions, as has been pointed out once more by Hilton in his recent article in the *Proceedings of the London Mathematical Society*, volume 5 (1907), page 1. A number of other known results are proved in this article without giving due references.

If a group contains one invariant subgroup of index $p$, the number of the invariant subgroups of this index is of the form $1 + p + p^2 + \cdots + p^\lambda$, $p$ being any prime number. In particular, the number of subgroups of half the order of any group is some number of the series 0, 1, 3, 7, 15, $\cdots$. The value of $\lambda$ in the preceding expression cannot exceed $m - 1$ when $p^m$ is the order of a Sylow subgroup of a group. If $p = 2$ and

* Burnside, *Proc. London Math. Society*, vol. 1 (1904), p. 388; also, vol. 35 (1903), p. 28.

† *Amer. Jour. of Mathematics*, vol. 27 (1905), p. 315.

‡ Fite, *Annals of Mathematics*, vol. 6 (1904), p. 7.

§ Zsigmondy, *Monatshefte für Math. und Physik*, vol. 7, (1896) p. 205

$m = 1$, $\lambda$ must have this maximal value. The possible groups of order $p^m$ when $\lambda = m - 2$ have been considered under the heading "Generalization of the hamiltonian groups" and the analogies between these groups and the hamiltonian groups have been studied. For smaller values of $\lambda$ the difficulty of the problem of determining all the possible groups greatly increases, so that the case when $\lambda = m - 3$ has not yet been completely solved.* Another generalization of the hamiltonian groups is due to Wendt and was published in the *Mathematische Annalen* almost simultaneously with the appearance of a paper dealing with the same question, in a somewhat more restricted manner, and bearing the title "The groups in which every subgroup of composite order is invariant."† The number of the possible invariant subgroups of index $p^2$ has also been studied but the results are less simple than those in regard to invariant subgroups of index $p$.‡ Several interesting general theorems have been deduced from the one which affirms that the number of cyclic subgroups of order $p^m$ in any group whose Sylow subgroups of order $p^\alpha$ are non-cyclic is divisible by $p$ whenever $p > 2$ and $m > 1$. One of these relates to the number of cyclic subgroups of any order $k = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\lambda^{\alpha_\lambda}$, $p_1 < p_2 < \cdots < p_\lambda$; $p_1, p_2, \cdots, p_\lambda$ being distinct prime numbers. If any group contains more than one cyclic subgroup of order $k$, it contains at least $p_1$ such subgroups, and if it contains exactly $p_1$ such subgroups, then $\alpha_1 > 1$ and it contains only one subgroup of order $p_\lambda^{\alpha_\lambda}$ $(\lambda > 1)$. In particular, there is no group which contains exactly two cyclic subgroups of the same order when this order is either odd or twice an odd number while there is an infinite number of different groups which contain exactly two cyclic subgroups whose order is an arbitrary number divisible by 4. A necessary and sufficient condition that a group is the direct product of its Sylow subgroups is that the $n$th power of each of its operators is contained in every subgroup of index $n$, for every possible value of $n$.§

Characteristic subgroups have the same property with respect to transformations under the group of isomorphisms as invariant subgroups have as regards transformations under the

* *Transactions Amer. Math. Society*, vol. 6 (1905), p. 326.

† BULLETIN, vol. 12 (1906), p. 379; *Archiv der Math. und Physik*, vol. 11 (1906), p. 76.

‡ *Comptes rendus*, vol. 140 (1905), p. 32.

§ *Amer. Math. Monthly*, vol. 13 (1906), p. 10.

group. As the group of isomorphisms is continually receiving more attention, so the characteristic subgroups are continually playing a more prominent rôle. The groups in which every subgroup is characteristic are cyclic, but all others with the exception of the elementary groups, that is, those which are either simple or the direct product of simply isomorphic simple groups, have both characteristic and non-characteristic subgroups. The most important characteristic subgroups are generated by the commutators of the groups which are not perfect. All the characteristic subgroups of an abelian group have been determined and it has been proved that they have a common subgroup known as the fundamental characteristic subgroup. The subgroup generated by the $i$th commutators is also characteristic.*

The characteristic subgroup generated by all the invariant operators of a group corresponds to the identity of the group of cogredient isomorphisms and hence it is of such unusual importance as to have received special names. In view of its dependence on this quotient group, it was first called the *cogredient subgroup*,† but in the text-book of de Séguier and in other French literature it has been denoted by the briefer term *the central*, while the corresponding quotient group has been called *the cogredient*. This quotient group is abelian in an important category of groups studied by Fite and others and known as metabelian groups.‡

The theorem that every group of order $p^m$ contains an abelian subgroup of order $p^a$ whenever $m > \frac{1}{2}a(a-1)$ has been extended by showing that the number of these abelian subgroups is of the form $1 + kp$ and hence at least one of them is invariant, and also by proving that every group of order 64 contains an abelian subgroup of order 16, while there are groups of order $p^6$ for all odd values of $p$ which do not contain an abelian subgroup of order $p^4$.§ In the Decennial Publications of the University of Chicago Moore determines all the subgroups of the generalized finite modular group. As noted in the preceding report, Wiman investigated the same problem, but Moore had practically completed his investigations before the appearance of Wiman's memoir.‖ The groups in which every two conjugate operators are commutative and those in

* Fite, *Transactions Amer. Math. Society*, vol. 7 (1906), p. 61.
† *Amer. Math. Monthly*, vol. 5 (1898), p. 221.
‡ Fite, *Transactions Amer. Math. Society*, vol. 7 (1902), p. 331.
§ *Messenger of Mathematics*, vol. 36 (1907), p. 188.
‖ Cf. *Jahrbuch über die Fortschritte der Mathematik*, vol. 34 (1905), p. 172.

which every pair of commutative operators, apart from identity, are conjugate have been investigated by Burnside and Rietz respectively.* It was found that the symmetric group of order 6 is the only group which satisfies the latter condition.

In volume 6 of the *Transactions* of this Society, Professors Moore, Dickson and Huntington extended their earlier results in regard to the definitions of an abstract group. Moore proves that one of the postulates in his earlier definition is redundant and establishes the mutual independence of the postulates in the new definition. Huntington gives a brief historical sketch of abstract definitions of groups and fields, and states that the first abstract definitions of fields are due to Dickson and himself, and are natural extensions of the sets of independent postulates which had already been given for groups. In various memoirs Dickson has developed a theory of groups in an arbitrary field defined by finite groups and has shown, in particular, that when the field is the general Galois field of order $p^n$ there is a doubly infinite system of finite groups which corresponds to each given finite group.† The work has close contact with the two earlier papers of Burnside, "On the continuous group that is defined by any given group of finite order."

If the number of operators of order 2 in an abelian group is at least equal to half the order of the group, then the group contains no operator whose order exceeds 2. There are, however, many different types of non-abelian groups in which the number of operators of order 2 is more than half the order of the group. The number of operators whose orders exceed 2 in such a group cannot be less than one fourth of the order of the group, and if it exceeds this number it must be at least equal to one third of the order of the group. For the groups of order $2^n$ all the possible ratios between the number of the operators whose orders exceed 2 and the order of the group have been determined on the hypothesis that this ratio is less than one half, and the maximum number of operators of order 2 in a group of any given order has been found.‡

In the enumeration of all the possible groups which satisfy given conditions encouraging progress has been made. As noted in my preceding report, the groups whose orders are the products

* Rietz, *Transactions Amer. Math. Society*, vol. 5 (1904), p. 500.

† Dickson, University of Chicago Decennial Publications, vol. 9 (1902), p. 35; *Transactions Amer. Math. Society*, vol. 3 (1902), p. 285.

‡ BULLETIN, vol. 13 (1907), p. 235.

of four prime numbers had all been considered with the exception of the case when the order is of the form $p^2qr$. The possible groups whose orders are of this form have recently been enumerated by Glenn.* The most extensive enumeration which has appeared during the period covered by the present report is that given in Potron's Paris University thesis. The author of this work aims to give a complete enumeration of all the possible groups of order $p^6$ together with some general types of groups of order $p^m$. Among the latter are those in which every subgroup of order $p^{m-2}$ is abelian while the group itself is metabelian. In a later note the author removes the latter restriction and makes some corrections of his enumeration of the groups of order $p^6$.† He uses the term *commutant* instead of the older terms derivative, or commutator subgroup.

Another extensive enumeration is due to Neikirk and is devoted to the group of order $p^m$ $(p > 2)$ which involve cyclic subgroups of order $p^{m-3}$.‡ The special case where $p = 2$ has been considered more recently by Miss McKelden.§ Among the less extensive enumerations which are not mentioned in connection with other subjects in the present report are the following: The groups which contain less than 15 operators of order 2 and are generated by these operators;‖ the groups of order $p^m$ which contain exactly $p$ cyclic subgroups of order $p^a$, those of order $2^m$ which contain an odd number of cyclic subgroups of composite order, and those which contain only three operators which are squares of other operators in the group.¶

All the operators of a group of composite order are contained in its cyclic subgroups but it is not always possible to select these subgroups in such a way that every two have only identity in common. The more general question of finding the properties of a group which satisfies the condition that all its operators are contained in a series of subgroups such that any two have only identity in common has been partially studied. The most interesting theorems that have been established may be stated as follows: If the group is non-abelian and of order $p^m$, then only one of these subgroups can involve operators of

* Glenn, *Transactions Amer. Math. Society*, vol. 7 (1906), p. 137.

† Potron, *Bulletin de la Société math. de France*, vol. 32 (1904), pp. 296, 300.

‡ Neikirk, *Publications of the University of Pennsylvania*, No. 3, 1905.

§ *Amer. Math. Monthly*, vol. 13 (1906), p. 121.

‖ *Amer. Jour. of Mathematics*, vol. 29 (1907), p. 1.

¶ *Transactions Amer. Math. Society*, vol. 7 (1906), pp. 94 and 238; vol. 6 (1905), p. 58.

order $p^2$. If it is abelian, its order must be of the form $p^m$ and its type must be $(1, 1, 1, \cdots)$.*

Dickson has found a new system of simple groups by the study for modulus 2 of a linear group on 7 variables which he had investigated before for fields not having this modulus. The problem for modulus 2 required a different analysis and led to a simple group of order $2^{6q}(2^{6q} - 1)(2^{2q} - 1)$, $q > 1$. For $q = 1$ the group has a simple subgroup of index 2 and of order 6048.† The following theorem is frequently useful in the study of simple groups, especially if they are represented as substitution groups. If $p^\alpha$ is the highest power of the prime $p$ which divides the order of a simple group $K$ and if $K$ contains less than $(p + 1)^2$ subgroups of order $p^\alpha$, then each of these Sylow subgroups is transformed into itself by a maximal subgroup of $K$.‡

In view of the fundamental importance of Sylow's theorem we add the following new statement of it, even though this statement can be readily derived from the one usually given. If the order of a group $G$ is $p^\alpha m$, $p$ being any prime which does not divide $m$, $G$ contains at least one subgroup $P_\alpha$ of order $p^\alpha$. Moreover if $P_\alpha$ contains only one subgroup $P_\beta$ of a given type, then all the subgroups of this type are conjugate under $G$ and their number is of the form $1 + kp$. The order of $G$ may be written in the form $p^\beta n(1 + kp)$, $p^\beta n$ being the order of the largest subgroup which transforms $P_\beta$ into itself.§ By letting $\beta = \alpha$ we obtain an ordinary form of this theorem.

(*To be continued.*)

---

## NOTES.

THE concluding (October) number of volume 29 of the *American Journal of Mathematics* contains the following papers: "On twisted quintic curves," by E. C. COLPITTS; "Attraction of the homogeneous spherical segment," by G. W. HILL; "On a certain class of algebraic translation surfaces," by J. EIESLAND; "Group characters of various types of linear groups," by H. E. JORDAN.

* BULLETIN, vol. 12 (1906), p. 446.
† Dickson, *Mathematische Annalen*, vol. 60 (1905), p. 137.
‡ *Comptes rendus*, vol. 136 (1903), p. 294.
§ *Annals of Mathematics*, vol. 5 (1904), p. 187.

THE fifty-seventh annual meeting of the American association for the advancement of science will be held at Chicago, December 30 to January 4, 1908, under the presidency of Professor E. L. NICHOLS, of Cornell University. Professor E. O. LOVETT is vice-president and chairman of section A (mathematics and astronomy) and Professor L. G. WELD is secretary. The address before Section A by the retiring vice-president, Professor EDWARD KASNER, will be on "Geometry and mechanics."

THE royal academy of sciences of Madrid announces the following prize problem for the year 1908:

"A concise exposition of the fundamental principles of nomography which are strictly necessary for the composition of a system of abaci or nomograms, as yet unknown and applicable with manifest advantage over any other procedure to the resolution of a series of questions, interesting in theory and useful in practice, referring to the physical and mathematical sciences."

The prize will be awarded in three classes. The first class, or prize proper, consists of a diploma of award, a gold medal, a money prize of 1500 pesetas, and a hundred reprints of the successful memoir for the author. The second prize consists of a diploma, a gold medal, and a hundred reprints. The third prize is a diploma. Competing memoirs must be written in Spanish or Latin and sent to the Secretaria de la Academia, 36 calle de Valverde, Madrid, before December 31, 1908.

THE late Dr. P. WOLFSKEHL, of Darmstadt, left the sum of 100,000 marks to the academy of sciences of Göttingen, to be awarded to the person who first presents a rigorous proof of the celebrated Fermat theorem: "The equation $x^n + y^n = z^n (n > 2)$ can never be satisfied by integers." Until this end has been attained, the interest on this sum is to be applied to mathematical purposes at the discretion of the academy.

THE time for submitting papers in competition for the Guccia prize (see BULLETIN, volume 11, page 167) expired July first. Three memoirs were received, bearing the titles: "Sur les courbes gauches de direction" (234 pages); "Sur quelques propriétés arithmétiques des courbes algébriques planes ou gauches" (16 pages); "Grundlage zu einer Bewegungstheorie des Kreises und der Kugel" (45 pages). On the sixth of July another memoir was received, bearing the title "Ueber die Uniformisierung beliebiger algebraischer Kurven" (70 pages),

but under the rules of the competition it could not be accepted. The result of the competition is to be announced at one of the sessions of the fourth international congress of mathematicians at Rome next April.

THE Hamburg mathematical society, founded in 1690, chiefly by the efforts of its first president, Heinrich Meissner, consisted at the time of its organization of six resident and nine non-resident members, and assumed the title "Kunstrechnungsübende Societät." That the society had a higher purpose than the solution of riddles is shown by the fact that it has continued in activity over two hundred years; it is the oldest mathematical society extant, and the oldest scientific society of Germany, except the Leopold-Carolus academy of Halle, which was founded in 1652. Its publications consisted at first of an annual letter or report and various subsidized books written by prominent members, the most important of which was the Mathematisches Sinnenconfect, written by P. Halcke in 1719. In the latter half of the eighteenth century over half the members resided in Holland; owing to the inconvenience of attending the monthly meetings at Hamburg, these members were largely influential in organizing the mathematical society of Amsterdam in 1778. From the time of the French revolution until the Franco-Prussian war, the chief activity of the society was the cultivation of applied mathematics, and most of its members were engineers, but since 1870 more interest has been shown in the broader view of mathematics. In 1873 an autographed abstract of the papers read before the society was appended to the yearly report and sent to all members. This proved so useful that it was continued until 1881, at which time the first number of the *Mitteilungen* was issued. The society now has over 60 members and possesses a substantial library. Although its influence has not been widely extended, yet the Hamburg mathematical society has been a most potent factor in maintaining a high standard of excellence in the schools of the city.

THE *Annuario* of the Circolo mathematico di Palermo issued in September, contains in addition to the usual alphabetic list of its members, a chronological list containing both former and present members distinguished by different type, a list of all the other learned societies to which any members belong, and the names of all such members in each, a similar list of univer-

sities and other institutions, and finally, a list of periodicals whose editorial staff includes one or more members of the Circolo. The society has 511 members, of whom 81 are Americans. During the year 1906 the membership increased by 107.

The Circolo has undertaken to publish the mathematical works and scientific correspondence of Paolo Ruffini, including the Theoria generale delle equazioni, which appeared at Bologna in 1799. The necessary funds have been generously provided by Professor G. B. GUCCIA, director of the *Rendiconti*.

The following university courses in mathematics are announced:

CAMBRIDGE UNIVERSITY. *Michaelmas term*, beginning October 14. — By Professor A. R. FORSYTH: Differential geometry, three hours. — By Professor G. H. DARWIN: Dynamical astronomy, three hours. — By Professor Sir R. S. BALL: Planetary theory, three hours. — By Professor J. LARMOR: Electricity and magnetism, three hours. — By Dr. E. W. HOBSON: Spherical harmonics and allied functions, three hours. — By Dr. H. F. BAKER: Introduction to the theory of functions, three hours, Theory of groups, three hours. — By Mr. B. A. HERMAN: Hydrodynamics, three hours. — By Mr. H. W. Richmond: Analytic geometry, three hours. — By Dr. A. N. WHITEHEAD: Principles of mathematics, three hours; Non-euclidean geometry, three hours. — By Mr E. W. BARNES: Linear differential equations, three hours. — By Mr. A. MUNRO: Hydrodynamics and sound, three hours. — By Mr. J. H. GRACE: Invariants and geometric applications, three hours. — *Lent term*, beginning January 16, 1908. — By Professor A. R. FORSYTH: Differential geometry, three hours. — By Professor G. H. DARWIN: Figure of the earth, three hours. — By Professor J. LARMOR: Electrodynamics with optical applications, three hours. — By Dr. E. W. HOBSON: Differential equations and expansions of mathematical physics, three hours. — By Dr. H. F. BAKER: Solid geometry, three hours; Theory of functions, three hours. — By Mr. B. A. HERMAN: Hydrodynamics, three hours. — By Mr. H. W. RICHMOND: Analytic geometry, three hours. — By Dr. A. N. WHITEHEAD: Principles of mathematics, three hours; Non-euclidean geometry, three hours. — By Mr. E. W. BARNES: Hypergeometric series, three hours. — By Mr. A. BERRY: Elliptic functions, Bessel functions and Fourier series, three hours. — By Mr. C. T. BENNETT: Line

geometry, three hours.— By Mr. A. YOUNG: Theory of invariants, three hours; Discontinuous groups, three hours. *Easter term*, beginning April 27, 1908.— By Professor A. R. FORSYTH: Differential geometry, three hours.— By Professor J. LARMOR: Theory of gases, three hours. — By Dr. H. F. BAKER: Theory of functions, three hours. — By Mr. B. A. Herman: Hydrodynamics, three hours. — By Mr. H. W. RICHMOND: Projective geometry, three hours.— By Dr. A. N. WHITEHEAD: Non-euclidean geometry, three hours.— By Mr. A. BERRY: Elliptic functions, three hours. — By Mr. A. MUNRO: Line geometry, three hours. — By Mr. G. H. HARDY: Integral functions, three hours.

UNIVERSITY OF VIENNA. *Winter Semester.* — By Professor G. v. ESCHERICH: Introduction to the theory of functions, five hours; Seminar, three hours. — By Professor F. MERTENS: Algebra, five hours; Seminar, three hours. — By Professor W. WIRTINGER: Calculus, five hours; Seminar, three hours. — By Professor G. KOHN: Analytic geometry, four hours; Differential geometry, two hours. — By Professor E. BLASCHKE: Introduction to the mathematics of statistics, three hours. — By Professor K. CARDA: Infinite groups, two hours. — By Dr. J. PLEMELJ: Theory of numbers, three hours. — By Dr. H. HAHN: Foundations of geometry, two hours. — By Dr. L. HANNI: Theoretic arithmetic, two hours — By Dr. L. SCHRUTKA: Finite discrete groups, two hours.

THE October number (volume 16, number 9) of the *Jahresbericht der deutschen Mathematiker-Vereinigung* contains an interesting abstract of the scheme of old age and disability pensions of full professors in the German universities and technical schools.

THE works of the Japanese mathematician SEKI, a contemporary of Newton, entitled Shichibusho (Seven Books), are soon to appear in one volume under the auspices of the Tokio Mathematico-Physical Society, in commemoration of the two hundredth anniversary of the death of this noted scholar.

BARON D. KIKUCHI, formerly professor of mathematics in the Imperial University of Tokio, returned to that city on September 14 after a trip around the world during which he gave a number of addresses in England. The baron is chairman of a committee which is about to undertake the work of compiling a history of native Japanese algebra.

THE following have been elected honorary members of the London mathematical society: Professor G. CASTELNUOVO, of the University of Rome, Dr. G. W. HILL, of New York, Professor C. JORDAN, of the College of France, Professor V. VOLTERRA, of the University of Rome.

THE following mathematicians have been decorated by the German emperor: Professor D. HILBERT, of the University of Göttingen, with the order of the crown of the third class; Professor W. KIEPERT, of the technical school of Hanover, with the order of the crown of the second class; Professor R. v. LILIENTHAL, of the University of Münster, with the order of the red eagle of the fourth class; Professor W. KILLING, of the University of Münster, with the order of the crown of the third class.

PROFESSOR A. R. FORSYTH, of the University of Cambridge, received the honorary degree of doctor of laws at the 25-year jubilee of the University of Liverpool.

DR. M. ABRAHAM, of the University of Göttingen, has been promoted to an associate professorship of mathematics.

DR. G. HERGLOTZ, of the University of Göttingen, has declined the call to the University of Freiburg, Switzerland.

DR. M. FRÉCHET, of the lycée at Besançon, has been appointed professor of special mathematics at the lycée at Nantes.

DR. G. NITALI has been appointed docent in mathematics at the University of Genoa.

PROFESSOR G. LAURICELLA, of the University of Catania, has been elected a corresponding member of the Accademia dei Lincei at Rome.

PROFESSOR D. A. MURRAY, of Dalhousie College, has been appointed to the chair of applied mathematics at McGill University, as successor to the late Professor G. H. Chandler.

MR. MURRAY MACNEILL, assistant professor of mathematics at McGill University, has been appointed professor of mathematics at Dalhousie College.

PROFESSOR E. H. MOORE, of the University of Chicago, will spend most of the present academic year in Italy, leaving New York about New Year's and returning in October.

PROFESSOR A. N. SKINNER, of the U. S. Naval Observatory, has retired under the age limit from his position as professor of mathematics. MR. H. L. RICE has been appointed his successor.

AT Brown University, Professor N. F. DAVIS has been granted a year's leave of absence. DR. R. G. D. RICHARDSON, recently of Yale University, has been appointed assistant professor of mathematics, and DR. H. H. CONOVER, also recently of Yale, has been appointed instructor in mathematics.

MR. FLOYD FIELD has been appointed junior professor of mathematics at the Georgia School of Technology.

DR. B. L. NEWKIRK, of the Lick Observatory, has been appointed assistant professor of mathematics and mechanics in the University of Minnesota.

AT Iowa College, Grinnell, Ia., Dr. R. B. MCCLENON has been promoted to an assistant professorship of mathematics; Mr. R. E. HAWLEY has been appointed instructor in mathematics.

MR. J. D. SUTER and MR. E. JORDAN have been appointed instructors in mathematics at Stanford University.

The following academic appointments are also announced: Mr. J. B. REYNOLDS, instructor in mathematics at Lehigh University; Mr. G. R. CLEMENTS, instructor in mathematics at Williams College; Mr. J. W. MITCHELL, instructor in mathematics at the Agricultural College of Texas; Mr. R. B. STONE, instructor in mathematics at Bowdoin College; Mr. — BREWSTER and Mr. — COOK, tutors in mathematics at the College of the City of New York; Miss M. E. WELLS has resumed her instructorship at Mount Holyoke College.

Catalogues of second-hand mathematical books: G. E. Stechert and Co., 129 West 20th Street, New York, catalogue 18, exact sciences, including journals, 98 pages. — A. Hermann, 6 rue de la Sorbonne, Paris, catalogue 89, journals, mathematics, astronomy and geodesy, 2064 titles; catalogue of theses, 100 entries in mathematics and astronomy.

# NEW PUBLICATIONS.

## I. HIGHER MATHEMATICS.

ACHITSCH (A.). Ein neues Integrationsverfahren. (Progr.) Pola, 1906. 8vo. 52 pp.

ADHÉMAR (R. D'). Les équations aux dérivées partielles à caractéristiques réelles. Paris, Gauthier-Villars, 1907. 8vo. 86 pp. Fr. 2.00

ANDRIESSEN (C.). Ueber Erzeugniss kongruenter Grundgebilde. (Diss.) Strassburg, 1906. 8vo. 45 pp.

APFELSTEDT (M.). Ueber eine Gattung von projektiven Transformationsgruppen in fünf Veränderlichen. (Diss.) Greifswald, 1906. 8vo. 75 pp.

BACHELET (C.). Cenni elementari sui limiti e sui numeri incommensurabili, ad uso delle scuole tecniche e normali. Torino, Cossone, 1907. 8vo. 22 pp.

BARBARIN (P.). La géométrie non-euclidienne. 2e édition. Paris, Gauthier-Villars, 1907. 8vo. 92 pp. Fr. 2.00

BÜCHEL (H.). Ueber ein nicht holonomes System: Die Rollbewegung einer Kugel in einer Kugelschale. (Diss.) Strassburg, 1906. 8vo. 40 pp.

BURKHARDT (H.). Vorlesungen über die Elemente der Differential- und Integralrechnung und ihre Anwendung zur Beschreibung von Naturerscheinungen. Leipzig, Teubner, 1907. 8vo. 11 + 252 pp. Cloth. M. 6.00

CANTOR (M.). Vorlesungen über Geschichte der Mathematik. Vol. IV. Von 1759 bis 1799. 2te Lieferung. Abschnitt XXI von E. Netto: Kombinatorik, Wahrscheinlichkeitsrechnung, Reihen, Imaginäres. Abschnitt XXII von V. Bobynin: Elementare Geometrie. Leipzig, Teubner, 1907. 8vo. Pp. 201–402. M. 5.60

CAPPILLERI (A.). Einführung in die Ausgleichungsrechnung. (Methode der kleinsten Quadrate.) Wien, 1907. 8vo. 138 pp. M. 4.00

CAROSI (E.). Sviluppo di funzioni in serie doppie trigonometriche. Foligno, Campi, 1907. 8vo. 79 pp.

CORNACCHIA (G.). Sulla congruenza $x^n - y^n = z^n$ mod. $p$. Tempio, Tortu, 1907. 8vo. 18 pp.

DANNACHER (S.). Zur Theorie der Funktionen des elliptischen Zylinders. Frauenfeld, 1906. 4to. 38 pp. M. 2.40

DEL RE (A.). Lezioni di algebra della logica, dettate nella r. università di Napoli. Napoli, r. Accademia delle scienze fisiche e matematiche, 1907. 8vo. 104 pp. L. 3.50

DORIA (G. A.). Studio sulle funzioni intere semplici. Bologna, Cuppini, 1907. 8vo. 30 pp.

DURHOLD (P.). Ueber einen Kreisbündel sechster Ordnung. (Diss.) Jena, 1907. 8vo. 57 pp.

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Publiée sous les auspices des académies des sciences de Göttingue, de Leipzig, de Munich et de Vienne avec la collaboration de nombreux savants. Edition française. Rédigée et publiée d'après l'édition allemande sous la direction de J. Molk. Tome I (1er volume): Arithmétique. Rédigé dans l'édition allemande sous la direction de F. Meyer. Fascicule 2 (pp. 161–328). Leipzig, Teubner, 1907. 8vo. M. 4.20

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band III: Geometrie, redigiert von F. Meyer. Teil 1, Heft 1: F. Enriques, Prinzipien der Geometrie; H. von Mangoldt, Die Begriffe "Linie" und "Fläche;" M. Dehn und P. Heegaard, Analysis Situs. 220 pp. Band IV: Mechanik, redigiert von F. Klein und C. H. Müller. Teil 2, II, Heft 1: C. H. Müller und A. Timpe, Die Grundgleichungen der mathematischen Elastizitätstheorie; O. Tedone, Allgemeine Theorie der mathematischen Elastizitätslehre. Leipzig, Teubner, 1907. 8vo. 124 pp.

ENDERS (M. A.). Ueber die Darstellung der Raumkurve vierter Ordnung vom Geschlecht 1 durch Thetafunktionen. (Diss.) Strassburg, 1907. 4to. 29 pp.

FOUËT (E. A.). Leçons élémentaires sur la théorie des fonctions analytiques. 2e édition entièrement refondue. 1er Tome: Les fonctions en général. Paris, Gauthier-Villars, 1907. 8vo. 13 + 113 pp. Fr. 3.50.

FRANZ (K.). Ueber Kreisschnitte auf Flächen zweiter Ordnung, speziell über Kreisschnitte auf dem Ellipsoid. (Progr.) Hamburg, 1907. 4to. 23 pp.

FROBENIUS (G.). Ueber einen Fundamentalsatz der Gruppentheorie. II. Berlin, 1907. 8vo. 10 pp.

GESZNER (E.). Ueber die Asymptotenkurven einer Schar Konoidflächen im allgemeinen und die des Cylindroids im besonderen. (Diss.) Münster, 1906. 8vo. 38 pp.

GRAF (J. H.). Der Basler Mathematiker Leonhard Euler. Bern, 1907. 8vo. 24 pp. M. 0.80

HAUSER (W.). Ueber Resultanten- und Discriminantenbildung in der Theorie der elliptischen Thetafunktionen. (Diss.) Erlangen, 1907. 4to. 40 pp.

HEMARDINQUER (C.). Notions de mathématiques supérieures (Calcul intégral et différentiel); licence ès sciences physiques, écoles techniques, constructeurs et praticiens. Avec une préface de C. A. Laisant. Paris, Paulin, 1907. 18mo. 6 + 144 pp. Fr. 3 00

HOBSON (E. W.). The theory of functions of a real variable and the theory of Fourier's series. Cambridge, University Press, 1907. 8vo. 788 pp. 21s.

HUEBNER. See REIM.

KERL (O.). Voranschläge der Genauigkeit beim trigonometrischen Punkteinschalten. (Diss.) Berlin, 1907. 8vo. 59 pp.

KLEIN (F.). See VORLESUNGEN, also ENCYKLOPÄDIE.

KOHLRAUSCH (F. L.). Einführung in die Differential- und Integralrechnung nebst Differentialgleichungen. Berlin, Springer, 1907. 8vo. 7 + 191 pp. Cloth. M. 6.80

LAISANT (C. A.). See HEMARDINQUER (C.).

LANGENKAMF (O.). Ueber Saccheris Untersuchungen des Parallelenaxioms. (Diss.) Münster, 1907. 8vo. 30 pp.

LONEY (S. L.). The elements of coordinate geometry for matriculation. London, Macmillan, 1907. 8vo. 182 pp. 3s. 6d.

MAGENER (A.). Anallagmatische Punktkoordinaten im Kugelgebüsch und ihre Anwendung auf die nichteuklidische Geometrie. (Diss.) Strassburg, 1907. 8vo. 56 pp.

MANGELSDORF (E.). Eine neue Abbildung des linearen Strahlencomplexes auf den Punktraum. (Diss.) Strassburg, 1906. 8vo. 32 pp.

MEYER (F.). See ENCYCLOPÉDIE, also ENCYKLOPÄDIE.

MINKOWSKI (H.). See VORLESUNGEN.

MOLK (J.). See ENCYCLOPÉDIE.

MONTEL (P.). Sur les suites infinies de fonctions. (Thèse.) Paris, Gauthier-Villars, 1907. 4to. 110 pp.

MÜLLER (C. H.). See ENCYKLOPÄDIE.

MÜLLER (H.) Einführung in die Differential- und Integralrechnung. Zum Gebrauch an höheren Schulen bearbeitet. (H. Müllers mathematisches Unterrichtswerk.) Leipzig, Teubner, 1907. 8vo. 5 + 38 pp. M. 1.50

NERNST (W.) und SCHÖNFLIES (A.). Einführung in die mathematische Behandlung der Naturwissenschaften. Kurzgefasstes Lehrbuch der Differential- und Integralrechnung mit besonderer Berücksichtigung der Chemie. 5te Auflage. München, Oldenbourg, 1907. 8vo. 12 + 371 pp. M. 12.50

PEDROTTI (L.). Introduzione al calcolo differenziale colla teorica dei massimi e minimi. Milano, Società ed. Sonzogno, 1907. 16mo. 62 pp. L. 0.15

PINCHERLE (S.). Algebra complementare. Parte II: Teoria delle equazioni. 2ª edizione. Milano, Hoepli, 1907. 16mo. 8 + 174 pp.

REHFELD (C.). Ueber Klassenbildung und Klassenzahl in algebraischen Funktionenkörpern. (Diss.) Strassburg, 1906. 8vo. 52 pp.

REICHEL (W.). Ueber trilineare alternierende Formen in sechs und sieben Veränderlichen und die durch sie definierten geometrischen Gebilde. (Diss.) Greifswald, 1907. 8vo. 59 pp.

REIM. Das regelmässige Dodekaeder und Ikosaeder in ihren wechselseitigen Beziehungen, nach Angaben von Dr. Huebner dargestellt. (Progr.) Schweidnitz, Heege, 1907. 8vo. 28 pp. M. 1.50.

ROELCKE (O.). Ueber die Bäcklundsche Transformation der Flächen konstanter Krümmung. (Diss.) Greifswald, 1907. 8vo. 41 pp.

RÖHRBORN. Potential und Anziehung eines verlängerten Rotationsellipsoids für den Fall, dass die Dichtigkeit gleich $k_0 + k_1xy + k_2xz + k_3yz$ ist. (Progr.) Hirschberg, 1907. 4to. 28 pp.

SCHIMMACK (R.). See VORLESUNGEN.

SCHÖNFLIES (A.). See NERNST (W.).

SCHÜLKE (A.). Behandlung der Differential- und Integralrechnung im Unterricht (Progr.). Königsberg, 1907. 8vo. 30 pp.

TANNENBERG (W. DE). Cours de géométrie analytique. 3e fascicule: Courbes planes en général (Courbes définies par leurs équations paramétriques). Paris, Vuibert, 1907. 4to. Pp. 89 to 181.

TOLEDO (L. O.). Introducción al estudio de las funciones de variable compleja. Madrid, Fortanet, 1907. 215 pp. P. 5.00

VOLTERRA (V.). Leçons sur l'intégration des équations différentielles aux dérivées partielles. Upsala, 1906. 4to. 83 pp.

VORLESUNGEN, mathematische, an der Universität Göttingen. Leipzig, Teubner, 1907. 8vo.

I. 1. Klein, F. Vorträge über den mathematischen Unterricht an den höheren Schulen. Bearbeitet von R. Schimmack. 1er Teil: Von der Organisation des mathematischen Unterrichts. 9 + 236 pp. Cloth. M. 5.00

II. Minkowski, H. Diophantische Approximationen. Eine Einführung in die Zahlentheorie. 8vo. 8 + 236 pp. Cloth. M. 8.00

VOSS (A.) Ueber Krümmung und konforme Transformation. München, 1907. 8vo.

WEISS (E.). Anzahlbestimmungen für das Strahlennetz (lineare Kongruenz). Breslau, 1907. 8vo. 64 pp. M. 2.40

WÜNSCHMANN (K.). Ueber Berührungsbedingungen bei Integralkurven von Differentialgleichungen. Greifswald, 1907. 8vo. 35 pp. M. 1.60

## II. ELEMENTARY MATHEMATICS.

BACHELET (C.). Problemi e teoremi di geometria euclidea. I. Torino, 1907. 8vo. 11 pp.

BAKER (W. M.) A key to algebraic geometry. London, Bell, 1907. 8vo. 7s. 6d.

BOTZHEIM (A. DE). La quadrature du cercle; essais géométriques. St. Blaise, 1906. 4to. 6 pp. Fr. 1.50

BOURLET (C.). Précis d'algèbre contenant 573 exercices et problèmes, rédigé conformément aux programmes du 31 mai 1902 et du 27 juillet 1905. (Classes de troisième B, seconde et première C et D.) 4e édition, revue et complétée. Paris, Hachette, 1907. 16mo. 443 pp. Fr. 2.50

BURRAN (C.). Tafeln der Funktionen Cosinus und Sinus mit den natürlichen sowohl reellen als rein imaginären Zahlen als Argument (Kreis- und Hyperbelfunktionen). Mit Einleitung in deutscher, französischer und englischer Sprache. Berlin, 1907. 8vo. 20 + 63 pp. Cloth. M. 4.00

CALVITTI (G.). Elementi di trigonometria piana e sferica ad uso delle scuole secondarie. Ascoli, Piceno, 1907. 8vo. 92 pp.

CELENTANO (A.). Nozioni di algebra elementare per le scuole tecniche. Napoli, Casella, 1907. 16mo. 50 pp. L. 1.00

COMSTOCK (C. E.). Elementary Algebra. Part I. Peoria, Ill., Comstock, 1907. 12mo. Cloth. $1.10

CRÜWELL (E. R.). Regeln der Mathematik. 3te, vermehrte Auflage. Lehrgang I: Planimetrie und Grundlehren der Stereometrie. Berlin, 1907. 8vo. 114 pp. M. 10.00

DICKMANN. See HEILERMANN, also KOPPE.

ENRIQUES (F.). Fragen der Elementargeometrie. Aufsätze von M. Amaldi, E. Baroni, E. Daniele, G. Vitali und anderen, gesammelt und zusammengestellt. Deutsche Ausgabe von H. Fleischer. (2 Teile.) Teil II: Die geometrischen Aufgaben, ihre Lösung und Lösbarkeit. Leipzig, 1907. 8vo. 12 + 348 pp. Cloth. M. 9.00.

FLEISCHER (H.). See ENRIQUES (F.).

GRAF (J. H.). Zur Geschichte der mathematischen Wissenschaften an der ehemaligen Akademie und der Hochschule Bern. Bern, 1906. 8vo. 19 pp.

GRÉVY (A.). Compléments de géométrie, à l'usage des élèves des classes de mathématiques A et B (Programme du 27 juillet 1905). Paris, Vuibert, 1907. 8vo. 477 pp. Fr. 2.00

HAASE (E.). See WILK (E.).

HALBGEBAUER (H.). See MOČNIK.

HARRIS (G.). Heuristic geometry for elementary schools. London, Meiklejohn, 1907. 8vo. 9d.

HAWKINS (C.). Elementary trigonometry. London, Dent, 1907. 8vo. 310 pp. 4s.
With answers, 4s. 6d.

HEILERMANN and DIEKMANN. Lehr- und Uebungsbuch für den Unterricht in der Algebra an den höheren Schulen. Neu bearbeitet von K. Knops. 1er Teil, 12te Auflage. Essen, Baedeker, 1907. 8vo. 8 + 236 pp. M. 2.25

HÖHM (F.). Geometrische Anschauungslehre für die I. bis IV. Klasse der Mädchen-Lyzeen. 2 Teile. Wien, Tempsky, 1907. 8vo. 95 pp. M. 1.80

HUNTINGTON (E. V.). Four-place tables of logarithms and trigonometric functions. Abridged edition. Cambridge, Harvard Coöperative Society, 1907. 8vo. 28 pp. $0.35

KNOPS (K.). See HEILERMANN, also KOPPE.

KOPPE und DIEKMANN. Geometrie zum Gebrauche an höheren Unterrichtsanstalten. Ausgabe für Reallehranstalten, 3er Teil, 3te Auflage, bearbeitet von K. Knops. Essen, Baedeker, 1907. 8vo. 7 + 288 pp. M. 3.20

LARA (F.). Descripción y uso de las tablas trigonométricas de Schrön, con varios ejercicios resueltos. Toledo, Gomez Ulenor, 1907. 57 pp. P. 1.50

LENNES (N. J.). See SLAUGHT (H. E.).

MATRICULATION mathematics papers: Being the papers in elementary mathematics set at the matriculation examination of the University of London from Jan. 1897 to Jan. 1907. London, Clive, 1907. 8vo. 124 pp. 1s. 6d.

MATRICULATION model answers: Mathematics, Sept. 1904 to June 1907. London, Clive, 1907. 8vo. 130 pp. 2s.

MOČNIK. Geometrie und geometrisches Zeichnen für Knaben-Bürgerschulen. Bearbeitet von H. Halbgebauer. Ausgegeb[illegible] 1 Bande. Wien, Tempsky, 1907. 8vo. 216 pp. M. 2.50

MODEL answers to intermediate pure mathematics. New edition. London, Clive, 1907. 8vo. 154 pp. 2s. 6d.

MOSBACHER (L.). Die Definitionen und Regeln der elementaren Algebra und ihre Anwendungen. Nürnberg, 1907. 8vo. 43 pp. M. 1.60

MÜLLER (H.). Die Mathematik auf den Gymnasien und Realschulen. Für den Unterricht dargestellt. 1er Teil: Die Unterstufe. (Lehraufgabe der Klassen Quarta bis Unter-Sekunda). Ausgabe B. Für reale Anstalten und Reformschulen. Leipzig, Teubner, 1907. 8vo. 8 + 200 pp. M. 2.20

OTTO (F.) und SIEMON (P.). Lehr- und Uebungsbuch der Arithmetik und Algebra für höhere Mädchenschulen. Leipzig, Hirt, 1907. 8vo. 196 pp. M. 2.00

ROBBINS (E. R.). Plane and solid geometry. New York, American Book Co., 1907. 8vo. 5 + 412 pp. Cloth. $1.25

——. Solid geometry. New York, American Book Co., 1907. 8vo. Pp. 15 + 251 — 409. Cloth. $0.75

ROMANO (F.). Un teorema sulla teoria delle progressioni aritmetiche di ordine *k*. Avola, Piazza, 1907. 16mo. 8 pp.

RUSSELL (J. W.). A sequel to elementary geometry, with numerous examples. London, Frowde, 1907. 8vo. 212 pp. 6s.

SALOMON (Mme A.). Leçons d'algèbre, à l'usage de l'enseignement secondaire des jeunes filles (Classes de quatrième et cinquième années) et des aspirantes au brevet supérieur. 4e édition. Paris, Vuibert, 1908. 16mo. 190 pp. Fr. 2.00

SANFELICI (G.). Calcolo tacheometrico semplificato. Tavole a graduazione centesimale e sessuagenerie, arricchite dei logaritmi dei numeri e delle funzioni trigonometriche. 2a edizione in Italiano e Francese. Milano, 1907. 4to. 270 pp. M. 10.50

SCHWERING (K.). Trigonometrie für höhere Lehranstalten. Nach den amtlichen Lehrvorschriften bearbeitet. 3te Auflage. Freiburg, Herder, 1907. 8vo. 7 + 55 pp. M. 1.30

SIEMON (P.). See OTTO (F.).

SLAUGHT (H. E.) and LENNES (N. J.). High school algebra, elementary course. Boston, Allyn and Bacon, 1907. 12mo. 12 + 297 pp. Cloth. $1.00

TORRENTS y MONNER (A.). Pequeñeces matematicas. Barcelona, 1907. 4to. 6 pp. M. 1.00

WEBER (H.) und WELLSTEIN (J.). Encyklopädie der Elementar-Mathematik. Ein Handbuch für Lehrer und Studierende. (In 3 Bänden.) 3ter Band: Angewandte Elementar-Mathematik. Bearbeitet von H. Weber, J. Wellstein und R. H. Weber. Leipzig, Teubner, 1907. 8vo. 13 + 666 pp. Cloth. M. 14.00

WEBER (R. H.). See WEBER (H.).

WELLSTEIN (J.). See WEBER (H.).

WILK (E.) und HAASE (E.). Anweisung zur Geometrie der Mittelschule. 1er Teil. Dresden, Bleyl, 1907. 8vo. 3 + 81 pp. M. 1. 85

## III. APPLIED MATHEMATICS.

BACHELET (C.). Alcune considerazioni sulla trattazione della meccanica independentemente dal postulato V di Euclide. I. Torino, 1907. 8vo. 15 pp.

BAKER (W. M.). A key to elementary dynamics. London, Bell, 1907. 8vo. 10s. 6d.

BANDINI (S.). Sui momenti quadratici di un sistema magnetico di punti. Padova, Prosperini, 1907. 8vo. 32 pp.

BAST (O. DE). Eléments du calcul et de la mesure des courants alternatifs. Paris, Béranger, 1907. 8vo. 197 pp.

BAZARD. Mécanique théorique. Cours professé aux écoles nationales d'arts et métiers (4 volumes). Vol. I. Paris, 1906. 8vo. 507 pp. M. 8.50

BORN (M.). Untersuchungen über die Stabilität der elastischen Linie in Ebene und Raum, unter verschiedenen Grenzbedingungen. (Diss). Göttingen, 1906. 8vo. 101 pp.

CAPPA (S.). Corso di idraulica pratica. Torino, Pasta, 1907. 8vo. 124 + 1055 pp. L. 32.00

CAROSI (E.). Vibrazioni di una lamina elastica rettangolare. Foligno, Campi, 1907. 8vo. 16 pp.

CATTANO (P.). Elementi di analisi infinitesimale, ad uso degli studenti di chimica. Padova, Draghi, 1907. 8vo. 7 + 132 pp. L. 2.50

CONTALDI (P.). Elementi di meccanica. Vol. III (Le macchine). Testo. Lezioni per gli allievi dell' istituto industriale delle Marche in Fermo. Fermo, Cooperativo, 1907. 8vo. 592 pp.

DUNKERLEY (S.). Hydraulics. Volume I: Hydraulic machinery. London, Longmans, 1907. 8vo. 352 pp. 10s. 6d.

FÖPPL (A.). Vorlesungen über technische Mechanik. (In 6 Bänden.) 5er Band: Die wichtigsten Lehren der höheren Elastizitätstheorie. Leipzig, Teubner, 1907. 8vo. 12 + 391 pp. Cloth. M. 10.00

GLEIJESES (M.). Su di una proprietà involutoria dei metacentri. Napoli, Trani, 1906. 8vo. 2 pp. L. 0.25

——. Equazione della superficie dei centri di galleggiamento di un cilindro qualunque per inclinazioni intorno ad assi paralleli alle generatrici. Napoli, Trani, 1906. 8vo. 15 pp.

HELMERT (F. R.). Die Ausgleichsrechnung nach der Methode der kleinsten Quadrate, mit Anwendungen auf die Geodäsie, die Physik und die Theorie der Messinstrumente. 2te Auflage. Leipzig, 1907. 8vo. 18 + 578 pp. Cloth. M. 15.00

JONAS (G.). Die Berechnung der Stromkurve eines Kondensatorkreises aus der Resonanzkurve eines mit ihm lose gekoppelten Sekundärkreises. (Diss.) Strassburg, 1907. 8vo. 20 pp.

SCHILL (R.). Maturitätsaufgaben aus der darstellenden Geometrie nebst vollständigen Lösungen. Für die oberen Klassen der Realschulen und verwandter Anstalten sowie für das Selbststudium zusammengestellt und gelöst. 2er Teil: Darstellung von Körpern mit Parallel- und Zentralstrahlenflächen, sowie regelmässigen Körpern samt ihren Schattenkonstruktionen. Wien, Deuticke, 1907. 8vo. 5 + 92 pp. M. 3.00

SUPLEE (H. H.). The mechanical engineer's reference book: a handbook of tables, formulas, and methods for engineers, students and draftsmen. 3d edition, revised and enlarged. Philadelphia, Lippincott, 1907. 16mo. 12 + 5 + 922 pp. Limp leather. $5.00